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Biographical sketches of *Ali Amir-Moéz* and of *Murray R. Spiegel* appeared in the May-June issue.

A MATHEMATICAL MUNCHAUSEN

Paul Schillo

At the time of this writing, ten years have elapsed since the untimely death of I. Neustadt. During this decade, the reform movement in mathematics, which he pioneered, has been marking time - almost as if it were afraid to proceed without a leader of his imposing stature. But, in order to imbue mathematics with Neustadtian methodology, what is needed most is not a leader with Neustadt's revolutionary zeal but a few more revolutionary discoveries like he was able to make.

A sufficient reason for the failure to discover new mathematical truths in the manner outlined by Neustadt is the lack of disciples in his school. Every reformation must expect the forces of reaction to have their innings, but, in this case, the game is being forfeited without contest. Most mathematicians have an erroneous conception of Neustadt's position, some have had no occasion to use his famous law, with the result that they give him little thought, and instances are known where otherwise mature mathematical scholars have failed to acquaint themselves with even the rudiments of his philosophy. Therefore, while this article is simply a brief account of the man and his major work, it is hoped that it might serve to correct some of the misconceptions of Neustadt's intent and so help to proselyte his program.

One notion has it that Neustadt sought to destroy mathematics. Nothing could be further from the truth; his orientation was positive at all times, for what he wanted to see was its reconstruction along modern lines. His sole aim was to make the structure of mathematics functional, to have it adopt scientific method and, thereafter, take its place among the modern sciences. Because of the cruel way in which hasty critics have distorted his progressivism, it is a wonder that he has not been accused of having suggested the tearing down of the cathedral of Notre Dame before the erection of the United Nations' building.

A good example of the way in which he has been misunderstood is the popular interpretaion of his facetious disproof of Malthus' Law. From the readily verifiable fact that two can live as cheaply as one, he proceeded to show by conventional mathematical methods that ten billion can live as cheaply as one. To naive critics, this might seem to be an attempt to discredit mathematics as a whole; actually, it was intended merely to show the absurdity of mathematical induction, a technique of which he strongly disapproved. He had no quarrel with fundamental, time-tested devices like addition and multiplication and used them frequently in his experiments.

It was not without justification that Neustadt came to the conclusion that mathematics was suffocating in its steadily worsening state of introversion. He felt that it had accepted the egocentric predicament centuries ago and had given up trying to extricate itself. To him, it was autopophagy, all this waste of cognitive energy in producing theorems, which were no more than reworded statements taken from the particular axioms that happened to be in vogue. "The present pace of mathematics is that of a snail," he once said. "And what is a snail? It's just a snake that has swallowed its tail" It was this irritation with all self-centered philosophical positions that goaded him on to his classic proof of the existence of a third personal reality,, the essence of which is found in his often-quoted remark: "Er iszt; daher ist er!" He was sure that mathematics had been feeding off its own fat for so long that it had become easy prey to the predatory natural sciences which picked at its bones like it was cold turkey. He envisioned its rebirth and hoped to teach it how to hunt for new truths in order to survive in a fiercely competitive age.

Unfortunately, the complete story of Isack Neustadt has yet to be written. The information about his early life is fragmentary. If he wrote anything prior to the formulation of the great law which bears his name, there is no record of it. Only a sketchy description of his formative years has been gleaned from the close-mouthed villagers of Dilldorf, Pennsylvania where he was born and raised. His buddies in the Grand Army of the Republic, in which he served as a private until he suffered a head wound at the battle of Gettysburg, have added little to the portrait, except to agree that he kept to himself and was affectionately referred to as "Pop".

Concerning his life in Dilldorf after the war, this much is known: The little indications of budding genius caused the villagers to conclude that he had been hexed, a consequence of having taken up arms and of having been encamped for a long time near the wicked city of Philadelphia. He began to read non-religious books, some of them in English and some of them concerned with the heinous heresy of science. Also, he showed no inclination toward physical labor and did nothing but pick a little fruit during the harvest in order to get money for books. It is to the lasting credit of the gentle people of Dilldorf that, despite their ignorance of Isack Neustadt's greatness, they saw to it that he did not want for anything essential to his survival. He was given shelter in the smithy and shared the victuals in the blacksmith's dinner pail. In fact, he lived more comfortably than any of his benefactors. But the need that these good people failed to satisfy was his greatest need, the hunger to be understood. The admirable patience, with which he endured those pangs throughout his early years, was to be given good use later when he had to face the hard fact that most mathematicians were simply Dilldorfers with academic degrees.

His escape from the subtle tyranny of his idyllic, native village was one of those remarkable little accidents which, taken all together, play the dominant role in the pageant of science. During the roaring twenties three gangsters out of Cicero, Illinois rubbed out a Wet lobbyist in Washington. The federal men soon knew who the hoodlums were as well as their devious getaway route via the backroads of five states. But, until they spoke to Isack Neustadt, none of the witnesses, who had seen the black limousine on its chicken-killing, wash-dusting journey through the hinterland, could help them refute its owner's claim that it had not been outside of the Chicago environs since its purchase. And it was only by a stroke of luck that they were able to contact Neustadt. For, due to their dislike of the government and its minions, the people of Dilldorf treated the investigators coldly; and those who spoke English maintained with righteous indignation that, during the fruit harvest none of them could possibly have been so idle as to gape at a "devil's chariot". But, just as the government men were about to drive off, a village wag made some remark about "der verdammte Faulpelz", which caused the dour Dilldorfers to laugh most heartily. Questioned about this outburst, an unusually helpful young interpreter explained that they were amused by the thought of one who might not have been so busily engaged at the time of the vehicle's passing, and that was Isack Neustadt.

As it happened, Neustadt had a vivid recollection of the incident. In keeping with his neighbors' opinion of him, he had been napping under an apple tree alongside the road. He told of having a strange dream about the war, a part of his life about which he had all but forgotten. A falling apple struck him on his temporal scar and he awoke in the same state of intense alertness as when he had regained consciousness in that gully at Gettysburg. It was at that moment that he saw the car pass by in a swirl of dust. He could not make out any of its details, not even its color, but its license plate seemed to be suspended motionlessly before him. It was an Illinois plate with the number 2-12-21-2. Even if he had not been endowed with a photographic memory for numbers, he would have remembered that particular one, because he had a profound respect for numerical palindromes.

Since his testimony was critical in the successful prosecution of the three torpedoes, he received a five thousand dollar reward which, after taxes, was reduced to $2,122.12 by a remarkable coincidence. However, by going to court and by accepting money for just remembering a number without doing any laborious reckoning, he became an anathema to the people of Dilldorf. Therefore, he hiked to Harrisburg and hopped a freight train on its empty run to Florida. He took up residence in a boathouse on the ocean side of Miami Beach, where, because of his frugality, he was able to live out his life comfortably on the reward money. It is a matter of public record that he never received remuneration for his many published works; money meant nothing to him.

In the stuffy refrigerator car which took him south, he began to contemplate his great law. Meditating on the events which had altered his life, he considered the cause and affect relationship that had to exist between the falling apple and the remarkable number 2-12-21-2. He thought of another falling apple in the past and of the man who had seen its significance. After all, was not the latter's name merely the anglicization of his own? A combination of his native mysticism and the semi-delirium, induced by the stifling atmosphere of the car, led him inevitably to the conclusion that he was the reincarnation of Isaac Newton and that there was something of great importance to be discovered in the number 2-12-21-2. Whether or not one is disposed to believe that there was anything supernatural about such an accumulation of coincidences, it must be agreed that the subsequent behavior of Isack Neustadt constitutes an excellent illustration of the James-Lange theory. For, even if he were not Isaac Newton, he acted enough like his prototype to make any distinction seem irrelevant.

Both literally and figuratively, it can be said that Neustadt's Law was incubated in the warm sand of the Atlantic Ocean, for, during his first year of peripatetic study on the beach, he wrote the number 1-12-21-2 over and over again on that golden slate. When the first burricane in his experience seemed about to rip the boat-house from its pilings, his most brilliant idea struck with greater intensity than the wind. He converted the dashes in 2-12-21-2 to minus signs and put plus signs before all but the leftmost of the other digits, obtaining 2 - 1 + 2 - 2 + 1 - 2, a finite series with zero sum. He knew that other six-term series had the property of adding to zero, but he was not to be dissuaded from the further consideration of this particular one.

Within a month from that fateful moment, he made the final alteration in his series, whence it became .2 - 1 + 2 - 2 + 1 - .2. which as the reader knows, is now called "Neustadt's basic series" or, by some writers, simply "Neustadt's base". According to his own account, the first arithmetic progression which he used was 1, 2, 3, 4, 5, 6. When he produced the derived series (or derivative) .2 - 2 + 6 - 8 + 5 - 1.2 by multiplying the successive terms in the base by the respective terms in the progression and observed that it too had a zero sum, he was so overwhelmed by emotion that he had to sit down beside a palm tree and there fell into a state of unconsciousness.

However, with the discovery that, for any six-term arithmetic progression, the derivative of his base had zero sum, his work had just begun. He was then faced with the herculean task of proving that his observation did indeed have the status of a mathematical law. This he accomplished in seventeen months without any outside assistance.

Before beginning the experiment which was eventually to prove his law, he took as his control group twenty zero sum bases different from .2 - 1 + 2 - 2 + 1 - .2 and listed two thousand distinct arithmetic progressions to be used on the bases. Among these progressions there were many like 2.71828, 3.14159, 3.56490, 3.98821, 4.41152, 4.83483 so it should be evident that he did not limit the generality of his results by selecting special progressions which would make his work easier. His ability to calculate mentally helped him to complete this experiment in a shorter period of time than it would have taken ten ordinary mathematicians. But, despite this demonstration that an exceptional person can undertake an immense mathematical project and, single-handedly, bring it to a successful conclusion, it ought to be clear that the search for mathematical truths can best be carried on in well-equipped laboratories by highly trained staffs. In the future machines and teamwork must be at the heart of mathematics or it shall forever remain an agglomerate of results from uncoordinated individual effort. No doubt, those years spent in solitary labor on his famous law led Neustadt to organize the first mathematical cooperative and to preach the doctrine of centrally controlled research as an alternative to the inherently feeble laissez faire mathematics. But his hopes that this small group would be the nucleus of a universal organization were based on the faulty assumption that others were as devoted to this single objective as himself. The cooperative lasted less than four months and did nothing constructive with its modest resources except to contribute to the Lemke-for-president campaign fund.

On the first day of his experiment to show that, for any six-term arithmetic progression, the derivative of .2 - 1 + 2 - 2 + 1 - .2 is itself zero sum, Neustadt had a fortunate accident. Using the arithmetic progression -3, -1, 1, 3, 5, 7 on his base, he had just obtained the zero sum derivative -.6 + 1 + 2 - 6 + 5 - 1.4, his 45th verification of the law without a single contradiction. He was getting tired and mistook this derivative for one of the bases in his control group. When he applied the progression -3, -1, 1, 3, 5, 7 to -.6 + 1 + 2 - 6 + 5 - 1.4 he found that its derivative 1.8 - 1 + 2 - 18 + 25 - 9.8 was zero sum, something that had not happened previously to a derivative of a base in his control group. And, the next morning, when he realized his mistake, he discovered something more amazing. By applying the progression 1.6, 1, .4, -.2, -.8, -1.4 to 1.8 - 1 + 2 - 18 + 25 - 9.8, a derivative of a derivative (or a second derivative) of his base, he obtained a third derivative with zero sum: 2.88 - 1 + .8 + 3.6 - 20 + 13.72. He immediately revised his program for the experiment in order to prove the law as it is formulated today: For any arithmetic progressions, all derivatives of any order of the base .2 - 1 + 2 - 2 + 1 - .2 are zero sum.

Right after this law was copyrighted on Flag Day in 1932, many mathematicians thought that it could be proved much more easily by their conventional devices than by Neustadt's scientific method. How wrong they were is evidenced by the fact that, despite numerous attempts, none of them have been able to come up with a deductive proof of it. Their sole contribution has been a forceful demonstration of the fact that the time Neustadt devoted to his successful project is negligible when compared with the time used in their futile efforts to belittle his methodology.

Now, unlike many mathematicians, Neustadt was no absolutist. He knew that the real numbers were phenomena like atoms and organic cells and that the laws which govern their behavior can be ascertained only to a degree. This degree of certainty, or truth value, of a known law must be quantified by assigning to it a number between zero and unity, commonly called its probability. He determined the probability of the truth of his law to be a number which differed from unity well beyond the thousandth decimal place, which is so close to certainty that, should it ever fail, one might do well to look for some catastrophic alteration in the universe. The manner in which he obtained this probability from his experimental data depended on a special application of the famous formula:

$$x = \frac{a + b^n}{n}$$

which Euler communicated to Diderot. As if anticipating the current popularity of electronic computers, he used the binary system because of its simplicity and, consequently, set $a = -2$ and $b = 2$. He let v be the number of experimental verifications of his law and c the number of contradictions. Then, n was made 1 or the vth prime according as v was or was not zero. After he evaluated the integer x, the required probability was determined by the empirical formula:

$$P = 2^{-c} - 2^{-2c-x}$$

Because his experiment yielded no contradictions of the law, this formula was reduced to the simpler formula:

$$P = 1 - 2^{-x}$$

and P approached 1 asymptotically as the experiment progressed and the number of verifications increased.

With the completion of this work, Neustadt did not rest on his laurels. But it is not the intention of this article to go into all that he did. It is well known that, as a champion of science, he led many crusades. Yet, the power of leadership failed to corrupt him and he retained throughout his life that golden streak of humility which is the banner of true genius. He was as cordial with the mountain folk of central Florida as with the polished aristocrats of suburban New York. So unobtrusive was he in his frequent contacts with people along the beach that, unless they had gone to Miami with the intention of meeting him, they were unlikely to realize how important a person he was. This aura of simplicity, which made him such an amiable man, influenced even his most profound thoughts. It enabled him to see that complex numbers were just ordinary numbers perceived by complicated people.

It is vain but interesting to contemplate what new truths Isack Neustadt might have uncovered had he not met with sudden death in his favorite retreat by that seaside coconut palm, under which he would sit by the hour like the great Gautama. Who knows what secret of the universe was locked up in that last message which he left in the sand? Perhaps someday a devoted disciple of his will look at that eleven term mathematical legacy:

$$.2 - 4 + 27 - 96 + 210 - 302.4 + 294 - 192 + 81 - 20 + 2.2$$

and will see its significance. But, as a sagacious humorist put it at a recent testimonial dinner, it should be hoped that this favored fellow will be lying under an apple tree and not a coconut palm when he gets his inspiration; otherwise, he might not live to develop it.

Buffalo, New York.

THE ISOPERIMETRIC PROBLEM IN THE PLANE

Glen E. Bredon

1. We shall consider here the isoperimetric problem for convex regions in the plane and its extension to more general sets. Part I is completely elementary as regards the methods used and presupposes next to nothing on the part of the reader. In it we prove the existence and uniqueness of the solution to the isoperimetric problem in the plane, restricted to convex regions. The uniqueness of a solution, if one exists, can quite easily be extended to regions bounded by a Jordan curve, but the existence becomes a little more difficult to show. In the second part we consider the existence proof for these more general regions, and also formulate and prove a very general isoperimetric theorem valid for arbitrary measurable sets in the plane. This part presupposes a very little knowledge of point set topology and measure theory, and a few statements are left unproved. Some of the places where the words "clearly" or "obviously" are used are admittedly actual gaps in the exposition. The author feels, however, that these gaps can be easily filled in by the intelligent reader by straightforward, if perhaps long, reasoning.

Part I.

2. A convex region means here a closed set R in the plane such that if p and q are in R, then the line segment joining them lies in R.

The isoperimetric theorem for convex regions in the plane says that among all convex regions in the plane of a given area A, the circular disc of area A, and only this disc, has minimum circumference. Clearly, this is the same as saying that the circular disc has the largest area among convex regions with fixed circumference, and again, these are equivalent to saying that for any convex region of area A and circumference C, $A \leq C^2/4\pi$, with equality if and only if the region is a circular disc. (A and C are understood to be finite.)

We note here for completeness that the length of the boundary of a convex region, and indeed, in a certain sense, the length of any curve, is defined as the least upper bound of the lengths of the inscribed polygons. Note that if C is finite then the lengths of all the chords, and hence of those from any particular point, are bounded and so the convex region is itself bounded

Lemma 1. If γ and γ' are convex curves (i.e. curves that enclose convex regions) with lengths C and C', and if γ surrounds γ' then $C \geq C'$.

Proof. By the definition of C' we may assume that γ' is a convex polygon. The proof in this case is effected by successively replacing arcs of γ by line segments which are just the sides of the polygon γ' produced until they meet γ (see fig. 1). This operation clearly decreases circumference and we finally arrive in the end at the circumference of γ', giving us our result.

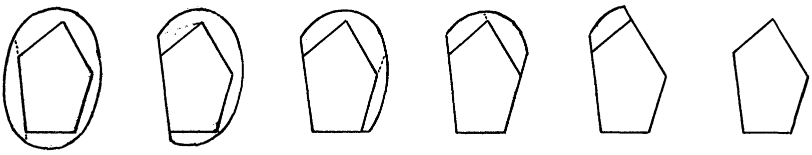

Figure 1

This lemma shows, among other things, that every bounded convex curve has a finite length. (That is, it is "rectifiable".) We need only enclose the curve in a suitable circle.

Next we develop briefly the interesting operation of symmetrization, due to Steiner. (See (5) and also Pólya (4).)

3. Given a bounded convex region R and a line l, the "axis of symmetrization", we define R_l the symmetrized image of R with respect to l as follows. Consider lines r which are perpendicular to l and cut R. Let r' be the segment of r which is the intersection of r with R. Let r_l be the segment of r which has the same length as r' and has its midpoint on l. We then define R_l as the union of all these line segments r_l. (See Fig. 2). It is easy to see that R_l is again convex.

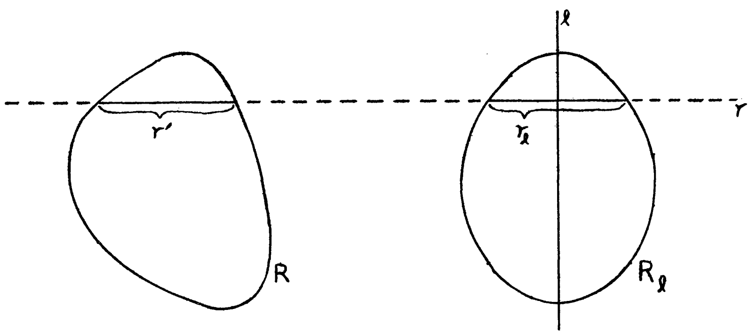

Figure 2

Lemma 2. Area (R_l) = Area (R) and Circ $(R_l) \le$ Circ (R).

Proof. That the area is invariant is an elementary application of integral calculus. We reduce the second statement to the corresponding one for polygons. Let P_l be a polygon symmetric to l and inscribed in R_l such that Circ $(P_l) \ge$ Circ $(R_l) - \epsilon$, where ϵ is an arbitrary positive number. Let P be the polygon inscribed in R whose vertices are mapped by the symmetrization into the vertices of P_l. P_l is then, in fact, the symmetrized image of P with respect to l. If the theorem is true for P and P_l then we have, using Lemma 1, Circ $(R_l) \le$ Circ $(P_l) + \epsilon \le$

Circ $(P) + \epsilon \leq$ Circ $(R) + \epsilon$ and, since ϵ is arbitrary, the theorem would be proved. The theorem for polygons clearly reduces to showing that the sum of the lengths of the lateral sides of a trapezoid with fixed bases and altitude is least when the trapezoid is isosceles, which in turn reduces to showing that among triangles with common base and given altitude the isosceles one has the least perimeter. This latter can be seen by use of an ellipse whose foci are at the ends of the base of the triangles and whose minor semi-axis is equal to the common altitude of the triangles. Since the sum of the distances from any point on an ellipse to the foci is constant we have (see fig. 3) $AC + CB \geq AD + DB = AE + EB$ giving our result.

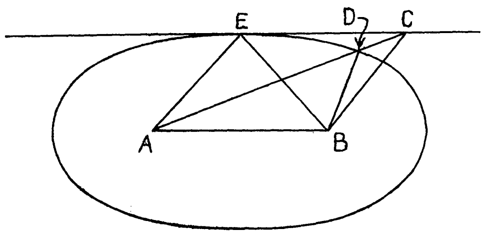

Figure 3

Lemma 3. If R is a bounded convex region and not a circular disc then there exists a centrally symmetric region R' which is convex, not a circular disc, of the same area as R and of no greater circumference than R.

Proof. We note first that any figure F, all of whose symmetrized images are circles, is itself a circle. If suffices to show that, for any n, a regular 2^n - gon can be inscribed in F. Select an arbitrary axis of symmetrization l and let P_1 be the chord of F which goes into the diameter of F_l perpendicular to l. This is our 2 - gon. Say that a 2^{n-1} - gon P_{n-1} has been inscribed in F. Then, given a side S of P_{n-1} consider the two symmetrizations in lines respectively perpendicular and parallel to S. The first shows there is a point of the boundary of F at the correct distance from S, and the second shows that this point must be precisely in the correct position to be a vertex of a regular 2^n - gon built on the vertices of P_{n-1}. This discussion is admittedly vague and the reader is urged to make a drawing and convince himself of the validity of the argument.

We now return to the region R in the statement of the present lemma. By the above we can assume that R is symmetric with respect to some axis l, since we can always find an axis of symmetrization which doesn't send R into a circle. Now divide R into two parts by this axis of symmetry and flip one side over as in figure 4. This clearly turns R into a centrally symmetric region with the same area and circumference, and which is convex and not a circular disc.

4. Proof of the Isoperimetric Theorem.

Theorem 1. (Uniqueness) If among all convex regions of circumference C, R has a maximum area, then R is a circular disc.

Proof. Using lemma 3, we know that if R is not a circle then there must be a convex, centrally symmetric region R' with the same area and circumference as R which is also not a circle. Let g be the greatest radius of R' and s the smallest. Then by using the construction indicated in Figure 5 we see that $s = g$, for, if not, we have obtained a region with area equal to $(g - s)^2 \sin \psi$ + area (R) and the same circumference as R. But the only centrally symmetric region with all radii equal is the circular disc, which shows that R must be a circular disc after all.

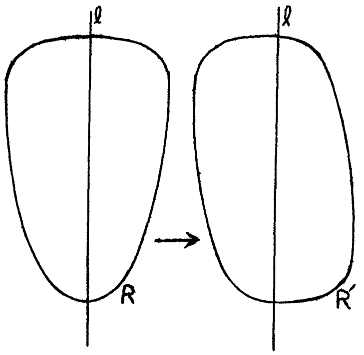

Figure 4

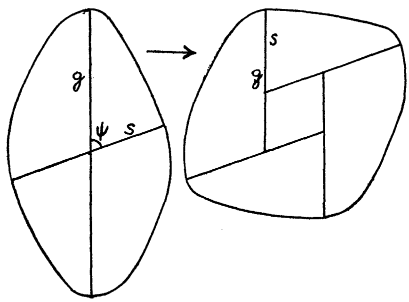

Figure 5

Theorem 2. (Existence) The circular disc has a maximum area among all convex regions with given circumference.

Proof. Say that R is a convex region with circumference C and area A such that $C^2/A < 4\pi$. We can assume R to be centrally symmetric by lemma 3 and hence can apply the operation indicated in Figure 5, which clearly gives a region which is again convex and centrally symmetric. Let R_n be the region which results from n applications of this operation and denote by g_n and s_n the greatest and smallest radii of R_n respectively and by ψ_n the acute angle between them. Each application of the transformation leaves the circumference C fixed and increases the area by $(g_n - s_n)^2 \sin \psi_n$. Since R_n is convex and s_n the smallest radius, R_n must lie completely to one side of a perpendicular to s_n at its endpoint. Hence we see that $s_n \geq g_n \cos \psi_n$ and thus

$$(g_n - s_n)^2 \sin \psi_n \geq (g_n - s_n)^3 / g_n$$

Since the area cannot increase indefinitely (it is bounded, for instance, by $(C/2)^2$), we must have $(g_n - s_n) \to 0$. Now consider two circles of radius s_n and g_n inscribedin and circumscribed about R_n respectively. Recalling lemma 1 we have the inequalities

$$2\pi s_n \le C \le 2\pi g_n \quad \text{and} \quad \pi s_n^2 \le A_n \le \pi g_n^2$$

which show that s_n and g_n approach a quantity r with the property that $C = 2\pi r$ and $\lim_n A_n = A' = \pi r^2$. But since $A_n \ge A$ we have that $4\pi = G^2/A' \le C^2/A < 4\pi$, a contradiction, proving the theorem.

Part II.

5. We have proved the isoperimetric theorem for convex regions. The uniqueness part of the theorem is easily seen to hold for regions bounded by a Jordan curve since any such region which is not convex clearly cannot be a solution to the problem. We will indicate two methods of extending the existence proof to these regions. Both methods are concerned with the so-called convex hull $h(R)$ of a region R. This is defined as the set of all points lying in triangles with vertices in R. When R is closed, $h(R)$ is convex and is clearly the smallest convex set containing R. Obviously the existence of a solution to the isoperimetric problem for the regions we are considering will follow if we can show that the circumference of $h(R)$ is not larger that that of R. (It is easy to see that the area is actually greater unless R is already convex.)

The first method of showing that circ $(h(R)) \le$ circ (R) lies in observing that the boundary of $h(R)$ consists of subarcs of the boundary B of R together with straight line segments joining points of B and also that these straight line segments replace arcs of B joining the same points. Anyone fairly familiar with point set topology should not have much trouble proving this statement and the existence proof of the isoperimetric problem follows immediately.

The second method rests on an interesting expression for the length of an arbitrary rectifiable curve, which we state as a theorem but will not prove, although the proof is not difficult (see Blaschka (1)).

Theorem 3. For any rectifiable curve γ with length L we have

$$(*) \qquad L = \tfrac{1}{2}\int_0^{\pi}\int_{-\infty}^{\infty} n(\rho,\psi)\, d\rho\, d\psi \qquad \text{(Lebesque)}$$

where $n(\rho,\psi)$ is the number of intersections (possiblyinfinite) of γ with a line of polar angle ψ and polar radius ρ.

Granting this we have immediately that circ $(h(R)) \le$ circ (R) since the corresponding $n(\rho,\psi)$ satisfy this inequality except possibly at the set of measure zero where the first of these is infinite.

6. By use of formula (*) we can state the isoperimetric problem for the plane in a very general form as follows. Given an arbitrary measurable set M in the plane, we define the circumference of M by the integral (*) where $n(\rho,\psi)$ is now the number of intersections of the line (ρ,ψ) with the boundary $\overline{M}\cap\overline{M'}$ of M ($'$ = complementation, $^{-}$= closure).

Theorem 4. If M is a measurable set in the plane with area (Lebesque measure) A and circumference C, then $A \le C^2/4\pi$. Furthermore, if $M = \overline{\text{int }(M)}$ (i.e. if M is the closure of its own interior) and C is finite, then equality holds only if M is a circular disc.

Proof. Since the inequality is trivial in case C is infinite, we will assume C to be finite. Given M we will make a series of replacements which do not decrease the isoperimetric quotient A/C^2 and will finally arrive at a convex set by these replacements, from which the first part of the theorem will follow by previous results.

i) Area $(\overline{M}) \ge$ Area (M) and $\overline{\overline{M}}\cap\overline{\overline{M}'}\subset\overline{M}\cap\overline{M'}$ so that Circ $(\overline{M}) \le$ Circ (M) We let $R = \overline{M}$.

ii) Let $K = \overline{\text{int }(R)}$. (Note that now $K = \overline{\text{int }(K)}$.) This operation is easily seen to be the result of dropping a certain set of boundary points, so that the circumference is certainly not increased. Say that the area is decreased. This implies that there is a set B of boundary points for which $m(B) > 0$ (Lebesque measure). Now for fixed ψ we have $m(B) = \int_{-\infty}^{\infty} m(\rho,\psi)\, d\rho$, where $m(\rho,\psi)$ is the linear Lebesque measure of the set of points in which B intersects the line (ρ,ψ). Since this is positive, the set of ρ for which B intersects the line (ρ,ψ) in an infinite number of points is of positive measure. Since this is true for all ψ, we have that C is infinite, contrary to assumption. Hence the area of K equals that of R.

iii) We define a relation $\sim$ on int (K) by: $p \sim q$ (p,q in int (K)) if and only if there is no k in K such that p and q lie in different components of $K - \{k\}$. This is clearly an equivilence relation on int (K) and since the equivilence classes are open sets there can be at most a countable number of them. Let $\{E_n\}$ be the set of these classes and set $F_n = \overline{E}_n$ and $F = \bigcup_n F_n$. Note that $K - F$ consists of boundary points alone and so by the argument under ii) does not contribute to the area. Now each F_n is met by the rest in at most a countable number of points and hence Area (K) = Area (F) = Σ Area (F_n) and Circ $(K) \ge$ Circ (F) = Σ Circ (F_n). Let

$$\alpha = \operatorname*{lub}_n \{\text{Area }(F_n)/(\text{Circ }(F_n))^2\}$$

(whose existence will follow later). Then we have

$$\text{Area }(K) = \text{Area }(F) = \Sigma \text{ Area }(F_n) \le \alpha\Sigma\,(\text{Circ }(F_n))^2 \le$$

$$\alpha\,(\Sigma \text{ Circ }(F_n))^2 = \alpha\,(\text{Circ }(F))^2 \le \alpha\,(\text{Circ }(K))^2$$

with strict inequality in the case that there are more than just one F_n. Hence there is an F_n such that

$$\frac{\text{Area }(F)}{(\text{Circ }(F))^2} \leq \frac{\text{Area }(F_n)}{(\text{Circ }(F_n))^2}$$

iv) Now consider a single F_n. We see that if any line meets F_n in exactly one point, then F_n must lie completely to one side of the line, and hence by formula (*) the circumference of the convex hull $h(F_n)$ is less than or equal to that of F_n. We have thus arrived at a convex set whose isoperimetric quotient is not less than that of M, and hence, by part I, we have proved the first part of our theorem. Previous remarks and the essence of the above proof render the second part of the the theorem obvious.

7. I take this opportunity to point out to the interested reader some of the different and highly elegant treatments of isoperimetric problems existing in the literature. Especially recommended is the excellent treatment by Pólya (3) and also that of Courant and Robbins (2). Most of the groundwork was done by Steiner (5), who, however, did not bother much with the question of the solutions existence. Some more advanced material is to be found in Blaschke (1), while Pólya (4) gives some highly interesting applications of symmetrization to mathematical physics.

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STABILITY OF PERODIC TIME-VARYING SYSTEMS

Louis A. Pipes

Summary

This paper presents a mathematical analysis of the stability of a class of mechanical systems that contain time-varying parameters of a periodic type. The mathematical analysis leads to the solution of linear second-order differential equations of the Mathieu-Hill type. For purposes of illustration the general methods presented are applied to the analysis of the motion of the "inverted pendulum".

I. *Introduction*

The general theory of oscillatory mechanical systems and their electrical analogues whose parameters are linear and constant has been extensively developed and is well understood. In recent years, considerable attention and effort has been directed to the analysis and to studies of the performance of dynamical systems whose parameters vary with the time. Special cases of technical importance in mechanical engineering are described in the treatises on mechanical vibrations by Timoshenko [1] and Den Hartog [2]. Time-varying systems of importance in electrical Engineering and communications include the induction generator, frequency-modulation circuits, the condenser microphone, and the super-regenerator. The theory of these circuits is summarized in the treatise by McLachlan [3].

The behavior of this entire class of dynamical systems can be described by differential equations that by suitable transformations can be reduced to an equation of the following general form,

$$\frac{d^2x}{dt^2} - F(t)x = 0 \tag{1.1}$$

where the function $F(t)$ is a single-valued periodic function of the time t of fundamental period T.

This second-order linear differential equation is known in the mathematical literature as a differential equation of the Mathieu-Hill type, [3]. The general solution of (1.1) depends on the nature of the periodic function $F(t)$. If this function can be represented by a Fourier series of the form,

$$F(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(nwt) + \sum_{n=1}^{\infty} B_n \sin(nwt) \tag{1.2}$$

where $w = 2\pi/T$, then (1.1) is known as Hill's equation. If, on the other hand, $F(t)$ can be represented by the first two terms of the Fourier series (1.2) so that it takes the form,

$$F(t) = A_0 + A_1 \cos(wt) \tag{1.13}$$

then (1.1) is known as Mathieu's differential equation.

The classical solution of Mathieu's differential equation leads to linear combinations of Mathieu functions [3]. The classical solution of Hill's equation involves computations with infinite determinants [3]. In general, the classical solutions are difficult to apply to special problems of practical importance.

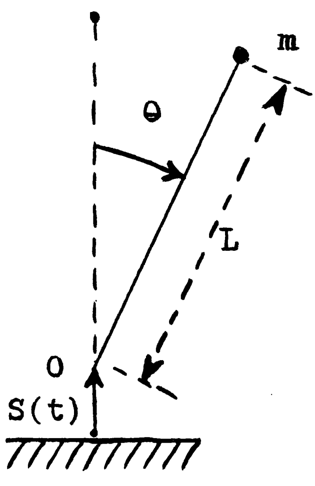
It is the purpose of this paper to describe two methods for the solution of equations of the type (1.1). One method involves matrix multiplication and the second method is an approximate one originally given by Lord Rayleigh, [4], but which has been used very little in the technical literature of vibrations. Although these methods are general, in order to provide a concrete physical example of their use, they will be applied to the study of the motion and stability of the "inverted pendulum," [2,5].

II. *Equations of Motion of the Inverted Pendulum.*

Consider the inverted pendulum of Figure 1. This pendulum consists of a weightless rod of length L to which is attached a mass m. The base of the pendulum is moved with a vertical acceleration $\ddot{S}$ by a crank or cam mechanism. It will be assumed that the pendulum undergoes a retarding frictional force at its pivot point 0 of a viscous friction type. If the angle θ that the pendulum makes with the vertical is small, it can be easily shown 2 that the equation of motion of the pendulum is

$$\ddot{\theta} + 2b\dot{\theta} - \frac{1}{L}(g + \ddot{S})\theta = 0 \tag{2.2}$$

The term $2b\dot{\theta}$ represents the retarding effect of the viscous frictional force of the prvot and g is the acceleration of gravity. In order to remove the first derivative term



Inverted Pendulum

Figure 1.

$\ddot{\theta}$ introduce the following transformation,

$$\frac{d^2x}{dt^2} - \left[\frac{1}{L}(g + \ddot{S}) + b^2\right] x = 0 \tag{2.3}$$

This differential equation is of the Mathieu-Hill type (1.1) with,

$$F(t) = \left[\frac{1}{L}(g + \ddot{S}) + b^2\right] \tag{2.4}$$

Two cases of the motion of the base of the pendulum will now be considered. In the first case, it will be assumed that by means of certain cam mechanism the base of the pendulum undergoes the following square-wave acceleration,

$$\begin{aligned} \ddot{S} &= K, & 0 \leqq t \leqq T/2 \\ \ddot{S} &= -K, & T/2 \leqq t \leqq T \end{aligned} \tag{2.5}$$

where K is a constant.

In the second case it will be assumed that through the action of a crank mechanism, the base of the pendulum is given the following harmonic motion.

$$S = a \cos(pt) \tag{2.6}$$

The effect of these two distinct types of motion of the base of the pendulum on its behavior will now be analyzed in detail.

III. *Motion of the Pendulum Produced by a Square-Wave Acceleration of its Base.*

In order to analyze this case, let,

$$k_1^2 = \frac{1}{L}(g + K) + b^2, \qquad k_2^2 = \frac{1}{L}(g - K) + b^2 \tag{3.1}$$

The differential equations of the motion of the pendulum (2.3) now take the form,

$$\ddot{x} - k_1^2 x = 0, \qquad 0 \leqq t \leqq T/2 \tag{3.2}$$

$$\ddot{x} - k_2^2 x = 0, \qquad T/2 \leqq t \leqq T \tag{3.3}$$

Let the initial angular displacement and angular velocity of the pendulum at $t = 0$ be θ_0 and w_0 respectively. These initial conditions in θ lead to the following initial valuesiin $x(t)$ by means of the transformation (2.2).

$$x(0) = \theta_0 = x_1(0) \tag{3.4}$$

$$\dot{x}(0) = (w_0 + b\,\theta_0) = x_2(0) \tag{3.5}$$

In order to determine the subsequent motion of the pendulum starting with the initial conditions (3.4) and (3.5) it is necessary to cascade the solutions of differential equations of the form (3.2) and (3.3) over intervals in time of duration $T/2$. This can be done by the use of a matrix multiplication scheme [6].

In order to do this, the solution of (3.2) may be written in the following matrix form,

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cosh(k_1 t) & \sinh(k_1 t)/k_1 \\ k_1 \sinh(k_1 t) & \cosh(k_1 t) \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{3.6}$$

with the notation $x_1 = x$ and $x_2 = \dot{x}$.

Let

$$\theta_1 = k_1 T/2 \tag{3.7}$$

then at the time t $T/2$, the solution of (3.2) may be written in the form,

$$\begin{bmatrix} x_1(T/2) \\ x_2(T/2) \end{bmatrix} = \begin{bmatrix} \cosh(\theta_1) & \sinh(\theta_1)/k_1 \\ k_1 \sinh(\theta_1) & \cosh(\theta_1) \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{3.8}$$

The solution of (3.3) may also be written in a matrix form similar to (3.6) with k_2 replacing k_1. The initial conditions of the solution of (3.3) are the values $x_1(T/2)$ and $x_2(T/2)$ given by (3.8). Hence the solutions for x_1 and x_2 at $t = T$ may be written in the convenient matrix form,

$$\begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} = \begin{bmatrix} C(\theta_2) & S(\theta_2)/k_2 \\ k_2 S(\theta_2) & C(\theta_2) \end{bmatrix} \cdot \begin{bmatrix} C(\theta_1) & S(\theta_1)/k_1 \\ k_1 S(\theta_1) & C(\theta_1) \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{3.9}$$

$$\theta = k_2 \frac{T}{2}, \quad C(\theta) = \cosh(\theta), \quad S(\theta) = \sinh(\theta) \tag{3.10}$$

If the indicated matrix multiplication is performed, (3.9) may be written in the following form,

$$\begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{3.11}$$

where,

$$A = \cosh(\theta_1)\cosh(\theta_1) + (k_1/k_2)\sinh(\theta_1)\sinh(\theta_2)$$

$$B = \sinh(\theta_1)\cosh(\theta_2)/k_1 + \sinh(\theta_2)\cosh(\theta_1)/k_2$$

$$C = k_2\cosh(\theta_1)\sinh(\theta_2) + k_1\sinh(\theta_1)\cosh(\theta_2)$$

$$D = \cosh(\theta_1)\cosh(\theta_2) + (k_2/k_1)\sinh(\theta_1)\sinh(\theta_2)$$

The solution at the time $t = nT$ or after n complete cycles of the variation of the acceleration of the base, can be obtained by casading the solution (3.11) n times. It is,

(3.13)
$$\begin{bmatrix} x_1(nT) \\ x_2(nT) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

As a consequence of the transofrmation (2.2), the angle of the pendulum at the time $t = nT$ is given by,

(3.14)
$$\theta(nT) = e^{-bnT} x_1(nT)$$

The stability of the pendulum therefore depends on the behavior of $x_1(nT)$ for large values of n. By means of Sylvester's theorem in matrix algebra [6], it is possible to write,

(3.15)
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^n = \begin{bmatrix} \frac{1}{S_1} & (S_{n+1} - DS_n) & (BS_n) \\ & (CS_n) & (S_{n+1} - AS_n) \end{bmatrix}$$

where

(3.16)
$$S_n = \sinh(an)$$

(3.17)
$$\cosh(a) = \frac{(A+D)}{2} = \cosh(\theta_1)\cosh(\theta_2) + [k_1^2 + k_2^2/2k_1k_2]\sinh(\theta_1)\sinh(\theta_2)$$

Therefore as a consequence of (3.12) and (3.14) the angle of the pendulum at $t = nT$ may be written in the following form,

(3.18)
$$\theta(nT) = \frac{e^{-bnT}}{s_1}[x_1(0)(S_{n+1} - DS_n) + x_2(0)BS_n]$$

Therefore it can be seen that for large values of n the angle θ will increase indefinitely and the pendulum will become *unstable* in the vertical position if,

(3.19)
$$a > bT, \quad \text{where } a = \cosh^{-1}\frac{(A+D)}{2}$$

If, on the other hand,

$$a < bT \tag{3.20}$$

the angle of the Pendulum θ will eventually approach zero and the pendulum will be *stable* in the vertical position. In the absence of damping, $b = 0$ and the vertical position will be an *unstable* one unless a is a pure imaginary number in which case the hyperbolic functions in (3.18) become trigonometric functions. In order for this to be possible, it is necessary that,

$$\left|\frac{A + D}{2}\right| < 1 \tag{3.21}$$

If (3.21) is satisfied, the pendulum will perform oscillations about the vertical position. Since k_1 is always positive, it is necessary that k_2 assume negative values in order for the condition (3.21) to be fulfilled. This means that the acceleration K must exceed that of gravity.

IV. *Harmonic Motion of the Base of the Pendulum.*

A study of the motion and stability of the inverted pendulum when its base is forced to execute a harmonic motion of the form,

$$S = a \cos (pt) \tag{4.1}$$

by the action of a crank mechanism will be undertaken in this section. The method of analysis is an approximate one suggested by Lord Rayleigh, [4].

If (4.1) is substituted into (2.1), it is seen that the equation of motion of the pendulum in this case takes the following form,

$$\ddot{\theta} + 2b\theta - \frac{1}{L}\,[g - ap \cos^2(pt)]\theta = 0 \tag{4.2}$$

To simplify the form of this equation, introduce the following change in variables,

$$\theta(t) = e^{-bt}y(t) \tag{4.3}$$

and

$$x = pt \tag{4.4}$$

In the new variables x and y, (4.2) takes the following form,

$$\frac{d^2y}{dx^2} - [(g/L + b^2)/p^2 - a \cos (x)/L]y = 0 \tag{4.5}$$

Now let,

$$w^2 = (g/L + b^2)/p \tag{4.6}$$

$$c = a/2L \tag{4.7}$$

Then (4.5) may be written in the convenient form,

$$\frac{d^2y}{dx^2} - [w^2 - 2\cos(x)]\, y = 0 \tag{4.8}$$

This is a form of Mathieu's differential equation and its general solution has the form 3,

$$y = \sum_{n=-\infty}^{n=+\infty} a_n e^{j(n+k)x} \tag{4.9}$$

where $j = (-1)^{\frac{1}{2}}$ and the a_n quantities are constants. k is a parameter to be determined by requiring (4.9) to be the solution of (4.8). If (4.9) is substituted into (4.8) and like powers of e^{jnx} are equated, the following set of homogeneous equations in the a's are obtained,

$$[(n+k)^2 + w^2]a_n - c(a_{n-1} + a_{n+1}) = 0 \tag{4.10}$$

where n runs through all values from − infinity to + infinity, and use has been made of the result,

$$2\cos(x) = e^{jx} + e^{-jx} \tag{4.11}$$

In order for the homogenous set of equations (4.10) to have other than the trivial solution $a_n = 0$, for all values of n, it is necessary for the determinant of the coefficients of (4.10) to vanish. The vanishing of this infinite determinant determines the value of the parameter k. The recursion formula (4.10) may then be used to determine the a's in terms of two arbitrary constants. This is an outline of the classical Floquet theory of the solution of Mathieu's equation [3].

If, however, the parameters of the pendulum are such that

$$w^2 << 1 \tag{4.12}$$

$$c^2 << 1 \tag{4.13}$$

as is usually the case, then the following approximate method for the solution of (4.8) given by Lord Rayleigh [4] may be used.

Instead of the infinite series (4.9) the following three term approximation for y is assumed,

$$y = \sum_{n=-1}^{n=+1} a_n e^{j(n+k)x} \tag{4.14}$$

This approximation is justified because as a consequence of (4.12) and (4.13) the values of the a's of higher indices are very small and can be neglected. As a consequence of the smallness of w^2 and c^2 it can also be shown [4] that the parameter k is also of the order,

$$k^2 << 1 \tag{4.15}$$

Hence the recursion formula (4.10) gives the three equations,

$$\begin{aligned} -a_{-1} + ca_0 &= 0 \\ ca_{-1} - (k^2 + w^2)a_0 + ca_1 &= 0 \\ ca_0 - a_1 &= 0 \end{aligned} \tag{4.16}$$

where use has been made of (4.12) and (4.15). In order for this system of homogeneous equations to have a non-trivial solution, the determinant of the coefficients of the a's must vanish. Hence,

$$\begin{vmatrix} -1 & c & 0 \\ c & -(k^2+w^2) & c \\ 0 & c & -1 \end{vmatrix} = 0 \tag{4.17}$$

If this determinant is expanded, the following equation for the parameter k is obtained

$$k^2 = 2c^2 - w^2 \tag{4.18}$$

If $2c^2 > w^2$, then (4.18) gives two *real roots* for the parameter k of the form,

$$k = \pm (2c^2 - w^2)^{\frac{1}{2}} \tag{4.19}$$

Equation (4.14) then gives the following two linearly independent solutions of (4.8),

$$\begin{aligned} y_1 &= C_1 e^{jkx} [ce^{-jx} + 1 + ce^{jx}] \\ y_2 &= C_2 e^{-jkx} [ce^{-jx} + 1 + ce^{jx}] \end{aligned} \tag{4.20}$$

where C_1 and C_2 are arbitrary constants. Since the equation (4.8) is linear, the general solution is the sum of these solutions and it can be written in the form,

$$y = [C_1 e^{jkx} + C_2 e^{-jkx}] \; [ce^{-jx} + 1 + ce^{jx}] \tag{4.21}$$

This can be written in the alternative form,

$$y = A \cos (kx + \phi) \; [1 + 2 \cos (x)] \tag{4.22}$$

where A and ϕ are two new arbitrary constants. In the absence of damping, $b = 0$ and the angle of the pendulum may be obtained from (4.3) and (4.22) in the form,

$$\phi(t) = A \cos (w_0 t + \phi) \left[1 + \frac{a}{L} \cos (pt)\right] \tag{4.23}$$

where

$$w_0 = \left[\frac{a^2 p^2}{2L^2} - \frac{g}{L}\right]^{\frac{1}{2}} \tag{2.24}$$

The condition of stability is that the parameter k be a real number and hence

$$2c^2 = w^2 \tag{4.25}$$

or

$$p > \frac{1}{a} (2gL)^{\frac{1}{2}} \tag{4.26}$$

That is, in order for the pendulum to execute stable oscillations of the form (4.23) about the vertical, the angular frequency of the oscillations of its base must satisfy the inequality (4.26).

If damping is included, then the pendulum will perform damped oscillations about the vertical position of equilibrium of the form,

$$\theta(t) = Ae^{-bt} \cos (w_0 t + \phi) \left[1 + \frac{a}{L} \cos (pt)\right] \tag{4.27}$$

provided,

$$p > \frac{1}{a} (2gL + L^2 b^2)^{\frac{1}{2}} \tag{4.28}$$

where

$$w_0 = \left[\frac{a^2 p^2}{2L^2} - \left(\frac{g}{L} + b^2\right)\right]^{\frac{1}{2}} \tag{4.29}$$

V. *Conclusion.*

The two methods presented in this paper for the analysis of dynamical systems whose equations of motion are of the Mathieu-Hill type are quite general and useful in cases where the classical theory is difficult to apply. The matrix multiplication method is particularly useful when the variable parameters are of the square-wave or saw-tooth type. The approximate Rayleigh method is a very useful one when it can be applied and seems to have been neglected in the technical literature.

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AN ELEMENTARY APPROACH TO THE USE OF THE RATE OF CHANGE CONCEPT FOR SOLVING PROBLEMS

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Foreword

This paper presents a method for deriving expressions for the rate of change of certain algebraic and trigonometric functions and for applying the rate of change concept to derive area and volume formulas without, at any stage of the development, introducing the use of limits.

From the time of the ancient Greeks all recognized methods of deriving such formulas, except for the simplest, require the use of limits in some form, ranging from the Greek method of exhaustion to the integral calculus.

The writer has not found, in searching through such writing on mathematics as were available to him, any indication that this method, or any other not involving the use of limits, has previously been used for solving the type of problems considered in this paper. Some statements were encountered to the effect that for specific problems no such method is known.

I. *Introduction*

A line which moves in a direction perpendicular to itself may be considered to generate an area by its motion, the rate of production of area is the product of the length of the line and its velocity, Fig. I.

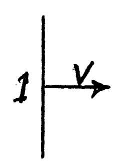


Fig. 1

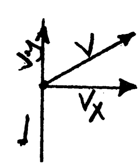


Fig. 2

If the line of motion is not perpendicular nor parallel to the line the velocity vector will have normal and parallel components, vx and vy Fig. 2. It is apparent that moving a line in a direction parallel to itself does not produce area, but only changes the position of the line. In Fig. 2 the area swept out by the motion of 1 in the direction of vector v in any time will be a parallelogram, the area of which is $1 \times$ altitude. The altitude will be proportional to vx, regardless of the value of v, for equal time of motion. Hence:

1. *The rate of area generation by the motion of a line is equal to the length of the line and its normal (perpendicular) velocity.* If the line is changing in length, one end will have a parallel velocity.

This velocity does not generate area, hence cannot affect the rate at which area is being generated at any specific time. It determines the rate at which the rate of area production is changing, assuming uniform normal velocity.

In what follows, all rates of change will be given in terms of one uniform velocity, which may be defined, without reference to limits, as change per unit time. If a velocity varies and its magnitude can be related to some other quantity, the expression showing the relationship sufficiently defines the velocity. Thus if $vy = 4xvx$, vy varies with x and vx and is specified as fully as is A in $A = \pi r^2$ for any value of r regardless of whether r is a constant or a variable.

In the problem to follow the proposition (1) given above and the following are the basis for most of the proofs:

2. If two quantities, originally zero, vary in such a manner that their respective rates of change are always in a constant ratio their magnitudes will always be in this ratio.

This is true for *any* uniform rates of change, hence must be true if the rates of change vary, for it is true for whatever rate of change the quantities may have at any instant and they can change only to another value for which it is also true.

In the notation used the rate of change of a quantity will be indicated by v, as vx, $v\theta$, etc; where coordinates are used vx will be assumed to be a uniform rate of change and vy will be expressed in terms of vx e.g. $vy = 3x\ vx$.

The area bounded by a curve, the x axis and the ordinate at x will be designated as Ax, similarly with respect to the y axis for Ay.

Volumes may be considered as generated by moving planes and the same arguments apply as given for generation of areas by lines. Hence:

3. The rate of volume production by a moving plane is the product of the area of the plane and its velocity perpendicular to the plane.

II. *Problems*

A. Consider the area Ax in the first quadrant, Fig. 3, under the line $y = x$. By geometry this area is known to be $\frac{1}{2}x^2$. This may be found by the rate of change method as follows:

$vAx = yvx$, which since $y = x$ is xvx

$vAy = xvy$, $vy = vx$, hence $vAy = xvx$

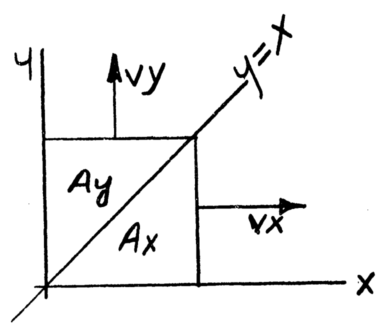


Figure 3

The total rate of area production is $xvx + xvx = 2\,xvx$ and the area produced is $xy = x^2$, hence $vx^2 = 2xvx$

$$\frac{Ax}{Ax + Ay} = \frac{vAx}{vAx + vAy} = \frac{xvx}{2xvx} = \frac{1}{2}$$

$$Ax = \tfrac{1}{2}(Ax + Ay) = \tfrac{1}{2}x^2$$

B. Next consider the area Ax in the first quadrant under the curve $y = x^2$. The total area A is $xy = xx^2 = x^3$.

$$vAx = yvx = x^2vx$$

$$vAy = xvy, \quad vy = vx^2 = 2xvx$$

from problem (A), hence,

$$vAy = x \quad xvx = x \quad vx$$

$$\frac{Ax}{A} = \frac{vAx}{vA} = \frac{x^2vx}{3x^2vx} = \frac{1}{3}$$

$$A = x^3$$

$$\therefore \quad Ax = 1/3\, x^3$$

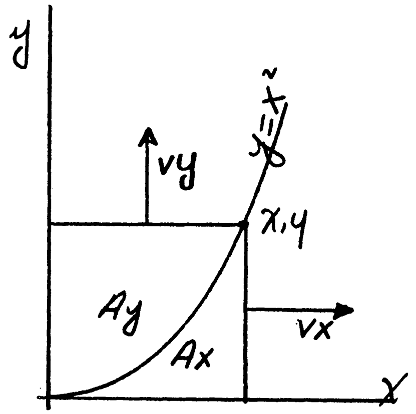


Figure 4

The formula has been derived for the area under the parabola $y = x^2$ and an expression obtained for the rate of change of x^3, namely $3x^2vx$. by following the same procedure as for $y = x^2$ the area under the curve $y = x^3$ is found to be $\frac{1}{4}x^4$ and the rate of change of x^4 found to be $4x^3vx$. Thus it can be shown that the rate of change and the area under any curve of the form $y = x^n$ are respectively $nx^{n-1}\,vx$ and $[1/(n+1)]\,x^{n+1}$ where n is any integer.

The same procedure may be used to show that the same formula applies when n is a fraction of the form $1/a$ where a is an integer.

C. The equation $y = \sqrt{x}$, Figure 5, is the inverse of $y = x^2$, hence $Ax = (2/3)y^3$. The same result may be obtained as follows: The rate of change of x^2 has been shown to be $2xvx$; x may be regarded as $(\sqrt{x})^2$. $vx = v(\sqrt{x})^2 = 2\sqrt{x}\; v\sqrt{x}$ from which

$$v\sqrt{x} = \frac{vx}{2\sqrt{x}} = vy$$

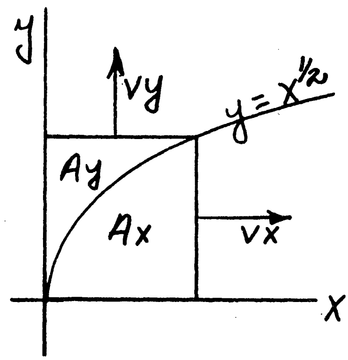


Figure 5

$vAx = yvx = \sqrt{x}\ vx$

$vAy = \sqrt{x}\ vy = \dfrac{xvx}{2\sqrt{x}} = \frac{1}{2}\sqrt{x}\ vx$

$vA = vAx + vAy = \sqrt{x}\ vx + \frac{1}{2}\sqrt{x}\ vx = (3/2)\sqrt{x}\ vx \qquad A = x\sqrt{x} = x^{1.5}$

$\dfrac{Ay}{A} = \dfrac{vAx}{vA} = \dfrac{\sqrt{x}\ vx}{(3/2)\sqrt{x}\ vx} = \dfrac{2}{3}$

$Ax = (2/3)A = (2/3)\ x^{1.5}$

Both $v(\sqrt{x})$ and the area under the curve are of the forms

$$vx^n = nx^{n-1}vx \quad \text{and} \quad Ax = \frac{1}{n+1}x^{n+1}$$

By following the same procedure for $y = x^{1/3}$, $y = x^{.25}$, and so on it can be shown that the same general equation is true for any fraction of the form $n = 1/a$ (a = integer).

To find the rate of change of a product, such as xy, construct a rectangle whose sides are x and y, Figure 6.

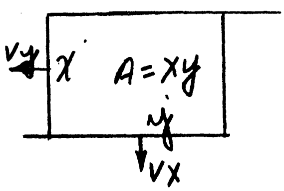


Figure 6

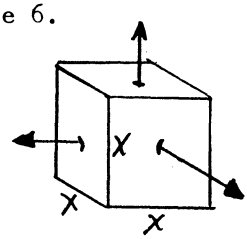


Figure 7

Then $vA = yvx + xvy = vxy$. If A is constant $yvx + xvy = 0$, since a constant has zero rate of change.

D. Find the formula for the volume of a pyramid, cone or any pyramidal prism, in terms of the area of the base and altitude.

The rate of change of a cube is $vx^3 = 3x^2vx$ (Fig. 7). The side x is increasing at a uniform rate.

If the side of a square increases at the same uniform rate as the cube in Fig. 7 and has a velocity vx perpendicular to its surface, the rate of volume production is x^2vx and the solid produced is a pyramid with base x^2 and altitude x. By giving the generating square the desired com-

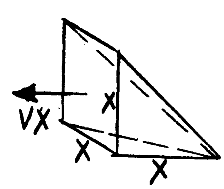


Figure 8

ponent of velocity parallel to its surface, the shape of the pyramid may be made regular or oblique as desired. A rotary motion in the plane of the square xx may be imparted, producing a spiral pyramid.

The rate of change of the pyramid, x^2vx, is 1/3 that of the cube, hence when x of the cube and pyramid are equal the volume of the pyramid is 1/3 that of the cube or $(1/3)x^3$.

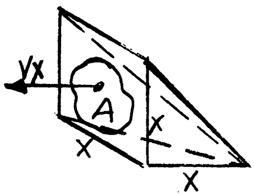


Figure 9

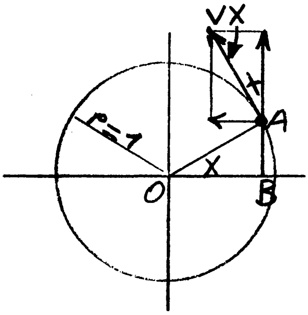


Figure 10

In Fig. 9 an area, A, is shown in the generating plane xx. This area has a constant ratio to the area of the square, that is A/x^2 = constant. Hence $\frac{vA}{vx^2} = \text{constant} = \frac{A\,vx}{x^2\,vx}$. The volumes produced will be in the same ratio. $\frac{\text{Volume of solid } A}{(1/3)x^3} = \frac{A}{x^2}$ from which, $\text{Volume of solid } A = \frac{A}{x^2} \cdot \frac{1}{3} x^3 = \frac{1}{3} A \cdot x$ where A is the area of the base and x is the altitude of the solid.

E. Find the formula for the rate of change of the sine, cosine, tangent, etc.

1. To find the rate of change of the sine and cosine construct, as shown in Fig. 10, a circle with radius 1, and tangential vector vx which represents the rate of change of the angle x. $AB = \sin x$ and $OB = \cos x$. The vertical component of vx is $\cos x\ vx$ and, since it is the velocity with which A is moving upward, it is the rate of change of AB. Hence $v \sin x = \cos xvx$. Similarly, the vector $\sin xvx$ = the rate of change of $\cos x$ and since $\cos x$ is decreasing this change is negative. $\therefore\ v \cos x = - \sin x\ vx$.

2. To find the rate of change of the tangent and secant, draw Figure 11. Here since $r = 1, AB = \tan x$ and $OA = \sec x$.

$$\frac{v_1}{vx} = \frac{OA}{1} = \sec x$$

$$v_1 = \sec x \; vx$$

The vector AC is the rate of change of the tangent since this is the velocity with which the length of AB is changing.

$$\frac{AC}{v_1} = \sec x \text{ or } AC = \sec x \; v_1$$

$$\text{but} \quad v_1 = \sec x \; vx$$

$$AC = \sec x \cdot \sec x \; vx = \sec^2 x vx$$

$$\therefore \quad v \tan x = \sec^2 x vx$$

v_2 is the rate of change of $\sec x$ (OA)

$$\frac{v_2}{v_1} = \tan x \text{ or } v_2 = \tan x \; v_1$$

$$v_2 = v \sec x = \tan x \cdot \sec x \; vx =$$

$$\sec x \cdot \tan xvx.$$

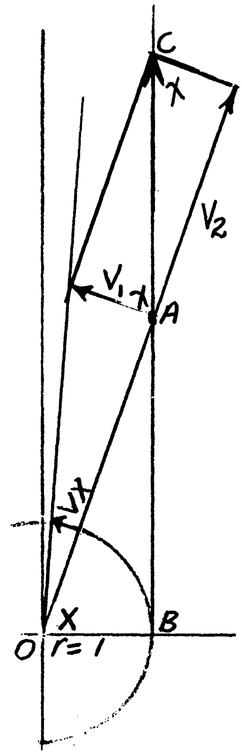


Figure 11

3. To find the rate of change of the cosecant and cotangent:

In Fig. 12,

$$\frac{v_1}{vx} = \frac{OA}{1} = \csc x$$

$$v_1 = \csc x \; vx$$

$$v_2 = \text{rate of change of } OA$$

$$\frac{v_2}{v_1} = \cot x$$

$$v2 = \cot x \; v_1 = \cot x \quad \sec x \; vx$$

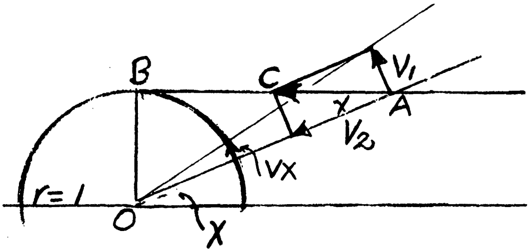


Figure 12

Since v_2 acts in a direction to decrease OA, the rate of change is negative.

$$\therefore \quad v \csc x = - \sec x \cot x \; vx$$

AC is the rate of change of AB and is negative because AB is decreasing.

$$\frac{AC}{v_1} = \csc x \text{ or } AC = \csc x \; v_1$$

$$v \cot x = -\csc x\, v_1 = -\csc x \csc x\, vx = -\csc^2 x\, vx$$

4. Find the rate of change of $\sin^2 x$.

In Fig. 13, make $OD = AB = \sin x$, then $DG = \sin^2 x$.

The point D has two components of velocity a normal, $DE = \cos x\, vx$ since the rate of change of $\sin x = \cos x vx$ and a tangential, $DH = \sin x vx$, since the tangential velocity is proportional to the distance from the center, O. The rate of change of $\sin^2 x$ is the vertical component of the resultant of the two velocities. This component is

$$KJ = KE + EJ$$

$$KE = DE \sin x = \cos x \sin x\, vx$$

$$EJ = DH \cos x = \cos x \sin x\, vx$$

$$KJ = 2 \sin x \cos x\, vx$$

This result can be obtained in a less complex fashion by considering a square whose sides are $\sin x$. The rate of change of the area, which is $\sin^2 x$ is $2 \sin x \cos x\, vx$.

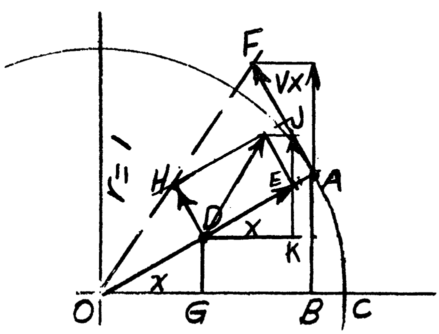


Figure 13

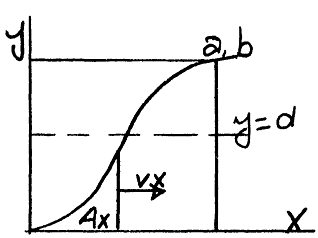


Figure 14

F. In the preceding derivations of volume and area formulas the ratio $\frac{yvx}{xvy}$ has been constant, with the result that the desired quantities were obtainable in terms of this ratio. When the quantities are not in a constant ratio to each other a modified procedure is necessary.

In Fig. 14 it is desired to find the area under the curve $y = f(x)$ between the values of $x = 0$ and $x = a$.

This area is equal to the area of some rectangle whose base is a and whose altitude is d, that is $Ax = ad$. d then is the average height of y in the interval under consideration. (Notation: $\overline{y}$ = average y.) The average value of vy is given by $\frac{\overline{vy}}{vx} = \frac{b}{a}$ or $\overline{vy} = \frac{b}{a} vx$. This does not directly assist in finding the required area though it does suggest a method of approach: if another equation, $y_1 = f_1(x)$, for which $vy_1 = f(x)(vx$ can be found, the average value of vy_1 will be the average height of y. Let $f_1(x) = c$ when $x = a$ and $f_1(x) = h$ when $x = 0$.

$$vy_1 = y\ \underset{-}{v}x$$

$$vy_1 = \frac{c-h}{a}\ vx = \bar{y}\ vx$$

$$y = \frac{c-h}{a}$$

$$A = \bar{y}\ a = \frac{c-h}{a}\ a = c - h$$

$$f_1(a) - f_1(0) = Ax$$

As an illustration of the method, find the area under any portion of one arch of the sine curve from $x = 0$ to $x = a$.

The rate of change of cos x is $-$ sin x vx hence $f_1(x) = -\cos x$.

$$A = -\cos a - (-\cos 0) = 1 - \cos a$$

If the equation of the curve contains more than one term the area under the curve will be the sum of the areas under the curve of each term considered separately, that is if $y = f_1(x) + f_2(x)$ the area under this curve will be $Axf_1(x) + Axf_2(x)$, since, Fig. 15, the rate of area production is the same whether the curves are separate or combined.

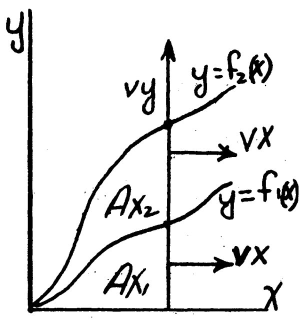


Figure 15

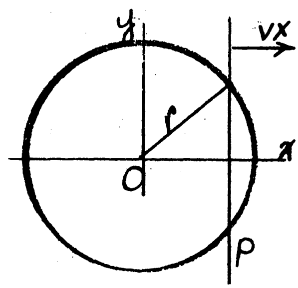
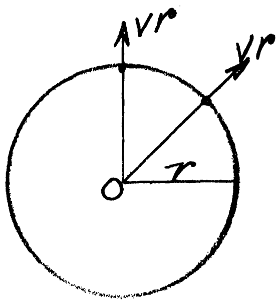


Figure 16

G. Find the formula for the volume of a sphere.

In Fig. 16 the sphere of radius r is cut by a plane p perpendicular to the x axis and moving with a velocity vx as shown. The rate of volume production is $\pi y^2 vx = \pi(r^2 - x^2)\ vx = \pi\ (r^2 vx - x^2 vx)$

and the volume of ½ the sphere $= \pi(r^2x - (1/3)x^3) = f_1(x)$ in the internal $x = 0$ to $x = r$, $V/2 = \pi(r^3 - (1/3)r^3) = (2/3)\pi r^3$, and $V = (4/3)\pi r^3$.



S = Surface
Q = Volume

Figure 17

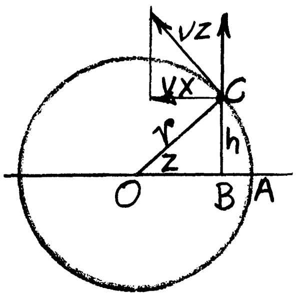


Figure 18

H. Find the surface of a sphere.

If the radius of a sphere, Fig. 17, is increasing at a rate vr, the rate of change of volume is $S\ vr$ since the entire surface is moving away from the center with a velocity vr.

$$vQ = S\ vr = v \cdot ((4/3)\pi r^3) = 4\pi r^2\ vr$$

$$\therefore\ S = 4\pi r^2$$

An alternate solution is as follows:

In Fig. 18 the sphere of radius r and center O is cut by a plane moving with a uniform velocity vx, with A taken as the origin. The surface considered will be that produced by the circle of radius h, at the intersection of the plane and the sphere, as the plane moves to the left in the figure. The rate of change of surface, $vS = 2\pi h\ vz$, where vz is the tangential velocity. $\dfrac{vz}{vx} = \csc z$, or $vz = \csc z\ vx$ and $h = r \sin z$.

$$\therefore\ 2\pi h\ vz = 2\pi r \sin z \cdot \csc z\ vx,$$

and

$$\sin z \cdot \csc z = 1. \quad \therefore\ vS = 2\pi r\ vx,$$

and the surface generated when vx has produced a length $2r$ is

$$2\pi r \cdot 2r = 4\pi r^2$$

This also indicates that the surface of the portion of the sphere between A and the cutting plane at C is that of the convex surface of a cylinder of radius r and altitude AB where

$$AB = r - \sqrt{r^2 - h^2}$$

The application of the method is that of integral calculus, except for the derivation of the formula for the areas under the curve $y = x^n$, which is direct, and yields the equivalent of the integral of the function.

The method of finding the equivalent of the derivatives of the trigonometric functions, which is unrelated to the differential calculus method, was discovered by the writer prior to his contact with differential calculus.

No attempt has been made to present this material in a form suittable for presentation to high school students, but rather to explain the basic principles involved and to demonstrate the possibilities of the method. Inverse trigonometric functions were not mentioned, since they are closely related to the direct functions. Logarithmic and exponential functions are not treated. No success has been attained in applying the velocity concept to the derivation of the rate of change of these functions.

NOTE: The integral of the sine and cosine are directly obtainable from Figure 10, $\sin x$ being the quantity produced by a velocity of $\cos xvx$ and $1 - \cos x$ is produced by a velocity $\sin xvx$ in the interval zero to x. By an extension of the method the areas under such curves as $\sin^2 x$, $\sin^3 x$, $x \sin x$, etc., may readily be found. This method was developed after the above paper was submitted.

Richland, Washington

MISCELLANEOUS NOTES

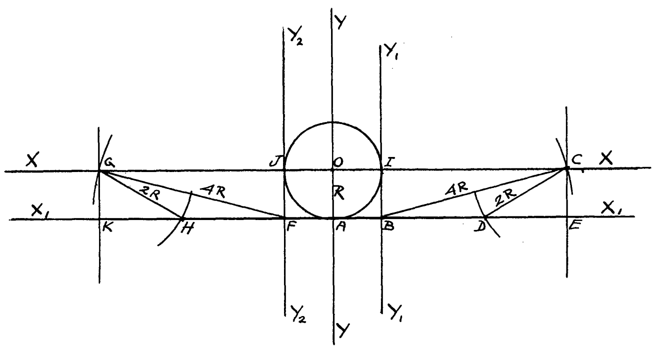
Edited by

Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

GRAPHICAL CONSTRUCTION TO FIND APPROXIMATE LENGTH OF THE CIRCUMFERANCE OF A CIRCLE

Jerzy Cicero-Pienkowski



Construction:

Draw a circle. Through its center draw two straight lines (xx) and (yy) perpendicular to each other. Draw straight line (x_1x_1) parallel to line (xx) and tangent to the circle. Also draw straight line (y_1y_1) parallel to (yy) and tangent to the circle. This will establish three points (A), (B), and (I), and the square, $(AOIB)$ whose sides are equal to R, the radius of the circle. From point (B) describe an arc with its radius equal to four times (OA), and intersecting line (xx). This will locate point (C). From point (C) describe an arc with its radius equal to twice (OA) and intersecting line (x_1x_1). Thus we shall get point (D). The distance from point (A) to point (D) will be approximately half of the circumferance of the circle. If we repeat the same construction on the left hand side of line (yy) we will get point (H) and since $(HA) = (AD)$

$$(HD) = \text{circumferance of the circle - approximately.}$$

Proof:

By construction

$$(yy) \perp (xx) \text{ and } (y_1y_1) \perp (x_1x_1)$$
$$(y_1y_1) \parallel (yy) \text{ and } (x_1x_1) \parallel (xx)$$
$$(BC) = 4(OA) = 4R$$
$$(CD) = 2(OA) = 2R$$

If we draw a straight line through point (C) and parallel to line (yy) then

$$(CE) = (OA) = R \quad \text{and} \quad \measuredangle(AEC) = 90^\circ$$

Also we get two right angled triangles: (BCE) and (DCE).

In $\Delta(BCE)$:

$$(BE)^2 = (BC)^2 - (CE)^2$$
$$(BE) = \sqrt{(BC)^2 - (CE)^2} = \sqrt{(4R)^2 - (R)^2} =$$
$$= \sqrt{16R^2 - R^2} = \sqrt{15R^2} = R\sqrt{15}\ ;$$

In $\Delta(DCE)$:;

$$(DE)^2 = (DC)^2 - (CE)^2$$
$$(DE) = \sqrt{(DC)^2 - (CE)^2} = \sqrt{(2R)^2 - R^2} =$$
$$= \sqrt{4R^2 - R^2} = \sqrt{3R^2} = \sqrt{3}\ .$$

$$(BD) = (BE) - (DE) = R\sqrt{15} - R\sqrt{3} = R(\sqrt{15} - \sqrt{3})$$

whence

$$(AD) = (AB) + (BD) = (OA) + (BD) = R + R(\sqrt{15} - \sqrt{3}) =$$
$$= R(1 + \sqrt{15} - \sqrt{3}) = R(3.14093\ldots.)$$

Now the circumferance of a circle equal $2\pi r$ where $\pi = 3.14159\ldots.$

Hence one half the circumferance equal $r \cdot 3.14159\ldots.$

Hence $(\overline{AD})$ equal one half circumferance of a circle - approximately; and (HD) equal the circumferance - approximately.

Buffalo, New York.

IBN HAITHAM'S PROBLEMS AND THEIR GEOMETRIC SOLUTIONS

A.R. Amir-Moéz

Ibn Haitham, (Alazen) (965-1039), has done a great deal in geometry and optics. He solved two problems by constructing the roots of quadratic or biquadratic equations. However, he was never satisfied with his solutions.

1. *Problem:* The circle (O) with center at O and the point P inside (O) are given. Find the direction of the ray through P such that after two reflections it passes again through P.

Solution: Suppose PT is the desired ray (Fig. 1). Clearly OT is the angle bisector of $\angle PTQ$ and TR, the tangent to (O) at T, is the angle bisector of $\angle MTQ$. Let $OP = a$, $OT = r$, $OQ = x$, $OR = y$. It is easy to see $r^2 = xy$ and $\frac{OQ}{OP} = \frac{RQ}{RP}$, i.e., $\frac{x}{a} = \frac{y - x}{y + a}$. Therefore we get $2ax^2 + r^2x - ar^2 = 0$. Clearly x can be constructed geometrically.

2. *The solution of Ibn Haitham's problem:* Suppose PT is the desired ray, (Fig. 2). Clearly TQ, the reflected ray, is perpendicular to OP. If TR is the tangent to (O) at T, then $T(POQR)$ is a harmonic pencil.

On the other hand TQ is the polar of R with respect to (O). Consider the line ML, through P and perpendicular to OP. The lines RM and RL intersect (O) respectively at C and B. BM and CL intersect at Q. $PQOR$ is a harmonic range, so is $LKBR$. Therefore MK passes through O. Consider the harmonic pencil $O(LKBR)$ which intersects LM at L, M, R, and X. Therefore $LPXM$ is a harmonic range and this shows the construction.

Ibn Haitham has solved the following problem by use of biquadratic. But the geometric construction of it is desired.

Problem: Given A and B on the plane of the circle (O) with center O, find the ray through A such that its reflection on (O) passes through B.

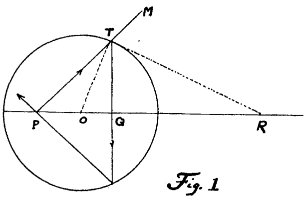


Fig. 1

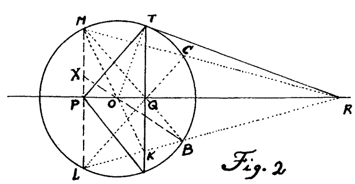


Fig. 2

University of Idaho.

On A Certain Problem In Mechanics

Murray R. Spiegel

In a book on mechanics, a problem similar to the following appeared.

> An object having mass of 1 gram moves in a straight line on a horizontal frictionless plane with initial velocity of 4 cm/sec. If it is acted upon only by a resisting force in dynes numerically equal to twice the instantaneous velocity in cm/sec, how far from the starting place does the object come to rest?

It was intended that the problem be solved as follows. Letting the object move on the x axis starting from the origin with initial velocity 4 cm/sec, it is clear that the differential equation for the instantaneous velocity is

$$(1) \qquad \frac{dv}{dt} = -2v$$

subject to the conditions

$$(2) \qquad v = 4 \quad \text{and} \quad x = 0 \quad \text{when} \quad t = 0$$

Since (1) may be written as

$$(3) \qquad v\frac{dv}{dx} = -2v \quad \text{or} \quad \frac{dv}{dx} = -2$$

we have on integration

$$(4) \qquad v = -2x + C$$

Applying conditions (2) it is seen that $C = 4$ so that

$$(5) \qquad v = 4 - 2x$$

Clearly then from (5) $v = 0$ when $x = 2$ and so the book states that the object comes to rest at a distance 2 cm from the starting place.

The above problem is slightly misleading in that it is but a simple matter to show that the object *never* comes to rest. To show this, start with equation (1) and obtain by separation of variables and use of conditions (2) the velocity

$$(6) \qquad v = 4e^{-2t}$$

which shows that v is never zero in any finite time and that as a consequence the object never comes to rest. The solution starting from equations (3) hides the time element in the problem and gives the false impression that the object does come to rest.

Rensselaer Polytechnic Institute.

A Proof of the Moment-Area Theorem

J. P. Hoyt

In finding the deflection of statically determinate beams with uniform cross-sections, the following ("moment-area") theorem is sometimes used:

> When a straight beam is subjected to bending, the distance (referred to as tangential deviation) of any point B on the elastic curve, measured normal to the original position of the beam, from a tangent drawn to the elastic curve at any other point A, is equal in magnitude to the product of $1/EI$ and the moment of the area bounded by the graph of the bending moment equation of the beam, the x-axis, and ordinates of the graph of the bending moment equation at A and B, with respect to the ordinate through B. (E is the modulus of elasticity of the material of which the beam is made and I is the moment of inertia of the area of the cross-section of the beam with respect to its neutral axis.)

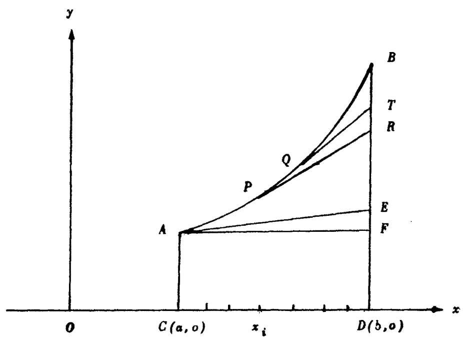


Figure I

In Figure I, let arc AB be the graph of $y = f(x)$, (where $f(x)$, $f'(x)$, and $f''(x)$ are continuous throughout the closed interval (a, b)) from $x = a$ to $x = b$. Let AE be tangent to AB at A and AF be parallel to CD. Since tangent $\angle EAF = f'(a)$, $EF = (b - a) f'(a)$, $FD = AC = f(a)$ and $BD = f(b)$. Hence $BE = BD - FD - EF$ or $BE = f(b) - f(a) - (b - a) f'(a)$.

Now although we have found the exact value of the tangential deviation, it is instructive as an illustration of summation by integration as well as leading to our desired result to find an integral whose evaluation will give BE.

To do this, divide CD into n equal (not necessary but sufficient) subdivisions of length Δx by the points x_1, x_2, x_3, ... x_{n-1}. Let P be the point on AB whose abscissa is x_i and Q be the point on AB

whose abscissa is x_{i+1}. Now $x_{i+1} = x_i + \Delta x$. Let PR be tangent to AB at P and QT be tangent to AB at Q.

By the formula for tangential deviation given in the preceding paragraph,

$$BR = f(b) - f(x_i) - (b - x_i)\, f'(x) \quad \text{and}$$

$$BT = f(b) - f(x_i + \Delta x) - (b - x_i - \Delta x)\, f\ (x_i + \Delta x)$$

so that

$$\Delta t_i = TR = BR - BT$$

$$= f(x_i + \Delta x) - f(x_i) - \Delta x f'(x_i + \Delta x) + (b - x_i)[f'(x_i + \Delta x) - f'(x_i)]$$

Now

$$BE = \sum_{i=0}^{n-1} \Delta t_i$$

for any n, so that

$$BE = \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta t_i$$

Since $f(x)$ and $f'(x)$ are continuous throughout (a, b), we can express $f(x_i + \Delta x)$ as $f(x_i) + f'(x_i)\Delta x + o(\Delta x)$ and $f'(x_i + \Delta x)$ as $f'(x_i) + f''(x_i)\Delta x + o(\Delta x)$ where $o(\Delta x)$ stands for a function of Δx whose ratio to Δx approaches 0 as Δx approaches 0.

Hence upon substituting and simplifying,

$$\Delta t_i = (b - x_i)\, f''(x_i)\Delta x + o(\Delta x)$$

And

$$BE = \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta t_i = \lim_{n\to\infty} \sum_{i=0}^{n-1} [(b - x_i)\, f''(x_i)\Delta x] + \lim_{n\to\infty} \sum_{i=0}^{n-1} o(\Delta x)$$

But

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} o(\Delta x) = 0.$$

So that

$$BE = \lim_{n\to\infty} \sum_{i=0}^{n-1} (b - x_i)\, f''(x_i)\Delta x = \int_a^b (b - x)\, f''(x)\, dx$$

by the fundamental theorem of the integral calculus.

This last integral can be recognized as the moment of the area bounded by $y = f''(x)$, $y = 0$, $x = a$, $x = b$, with respect to the line

$x = b$. Since the second derivative with respect to x of y in the equation of the elastic curve $y = g(x)$ is $1/EI$ times M_x where M_x is the the bending moment equation of the beam, the moment-area theorem follows.

Apart from the usefulness of this theorem in the strength of materials, it merits inclusion in calculus texts (I have never seen it in one) for two reasons: one, in deriving it we use the integration process to find the limit of a sum whose limit we already know: two, it can also be used to find not-so-easily calculuable moments from easily calculable tangential deviations. Such problems are also useful to an understanding of the theorem. One such example would be: Find the moment of the area lying in the first quadrant bounded by the lines $y = \sin x$, $y = 0$, $x = \pi/2$. This moment would be given by the integral

$$\int_0^{\pi/2} (\pi/2 - x) \sin x dx = \frac{\pi}{2} - 1.$$

This involves integration by parts. To apply the moment area-theorem, find the simplest function, $-\sin x$, whose second derivative with respect to x is $\sin x$. Draw the tangent to the graph of $y = -\sin x$ at $x = 0$, and find the tangential deviation at $x = \pi/2$. This gives the required moment. From figure II, the steps are obvious and simple.

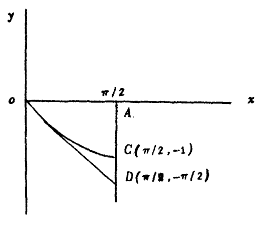


Figure II

$$\tan \angle DOA = -\cos 0 = -1$$

$$\therefore AD = -OA = -\frac{\pi}{2}$$

$$CD = AC - AD = \frac{\pi}{2} - 1$$

U.S. Naval Academy.

CURRENT PAPERS AND BOOKS

Edited by

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

* * *

Principles and Techniques of Applied Mathematics. By Bernard Friedman. John Wiley & Sons, 440 Fourth Ave. New York 16, N.Y. 315 pages, $8.00.

A unique combination of abstract theory and practical application marks this book. It is the latest addition to Wiley's Applied Mathematics Series, edited by I.S. Sokolnikoff.

Deploring the widening gap between pure and applied mathematics, Dr. Friedman states his belief that this gap is an illusion. He contends that the study of abstract systems helps solve concrete problems, and that the study of specific problems may, in turn, suggest interesting generalizations to the pure mathematician. *Principles and Techniques of Applied Mathematics* therefore sets out to show how the powerful methods developed by the abstract studies can be used to systematize the methods and techniques for solving problems in applied mathematics.

Two main themes carry out Dr. Friedman's approach. The first is a demonstration of how the abstract theory of linear operators can be used to unify the techniques of applied mathematics. Secondly, Dr. Friedman develops and explains specific techniques that can be used to obtain explicit solutions of partial differential equations.

Among the novel features appearing in this volume are a presentation of the Schwartz theory of distributions and the S-function with applications; a simple proof for the reduction of a matrix to Jordan canonical form, with the proof using generalized eigenvectors; a simple technique for obtaining the complete eigenfunction expansion for a second-order ordinary differential operator; a formal method for solving partial differential equations which includes all transform methods such as the Laplace and Fourier transforms; and the use of the general theory of the linear operators to obtain the Fredholm theory of integral equations. Admitting that his emphasis is on the techniques, Dr. Friedman maintains that a logical analysis of a problem leads automatically to the proper methods for its solution.

Richard Cook

Symposium on Monte Carlo Methods. Edited by H. A. Meyer. John Wiley & Sons, 440 4th Ave., New York 16, NY. 382 pages, $7.50.

The first full-lenth treatment of this subject, the new book is the work of twenty-two leaders in this promising field, writing about their own research and applications.

The volume is the outgrowth of the symposium conducted by the Statistical Laboratory of the University of Florida, and sponsored by the Wright Air Development Center. It is the latest addition to the Wiley Publications in Statistics, Walter A. Shewhart and S. S. Wilks Editors.

"The possibility of going from a physical process directly to a numerical solution without benefit of an intermediary differential equation," Dr. Meyer states, "is such an unorthodox and revolutionary approach that the imagination is kindled to reconsideration of unsolved problems of old."

Originally refined for solving atomic energy problems, Monte Carlo methods have begun to spread into other specialties such as market analysis, quality control, public-opinion polling, and genetics.

The papers appearing in this book range from the simplest to the highly theoretical and include the following: *An Introductory Note* by A.W. Marshall; *Generation of Pseudo Random Numbers* by Olga Taussky and John Todd; *Phase Shifts - Middle Squares - Wave Equations* by N. Metropolis; *A General Theory of Stochastic Estimates of the Neumann Series for the Solution of Certain Fredholm Integral Equations* by G.E. Albert; *Neighbor Sets for Random Walks and Difference Equations* by Theodore S. Motzkin; *Monte Carlo Computations* by Nancy M. Dismuke; *Applications of Monte Carlo Methods to Tactical Games* by S. Ulam; *Conditional Monte Carlo for Normal Samples* by Hale F. Trotter and John W. Tukey; *Monte Carlo Techniques in a Complex Problem about Normal Samples* by Harvey J. Arnold, Bradley D. Bucher, Hale F. Trotter, and John W. Tukey; and *An Application of the Monte Carlo Method to a Problem in Gamma Ray Diffusion* by Martin J. Berger.

The balance of the papers include: *Stochastic Calculations of Gamma Ray Diffusion* by L.A. Beach and R.B. Theus; *Application of Multiple Stage Sampling Procedures to Monte Carlo Problems* by A. W. Marshall; *Questionable Usefulness of Variance for Measuring estimate accuracy in Monte Carlo Importance Sampling Problems* by John E. Walsh; *Experimental Determination of Eigenvalues and Dynamic Influence Cooefficients for Complex Structures such as Airplanes* by C. W. Vickery; *Use of Different Monte Carlo Sampling Techniques* by Herman Kahn; *A Theoretical*

Comparison of the Efficiencies of Two Classical Methods and a Monte Carlo Method for Computing One Component of the Solution of a Set of Linear Algebraic Equations by J. H. Curtiss; *A Description of the Generation and Testing of a Set of Random Normal Deviates* by E. J. Lytle, Jr.; *Machine Sampling from Given Probability Distributions* by James W. Butler; *A Monte Carlo Technique for Obtaining Tests and Confidence Intervals for Insurance Mortality Rates* by John E. Walsh; and *Experiments and Models for the Monte Carlo Method* by Alwin Walther.

In addition, an extensive bibliography gives abstracts and summaries of existing literature, many of them formulated in the authors' own words. A considerable portion of this literature was previously available only in the form of government technical reports which were frequently of limited distribution. The appendix also contains a section on random digits, a section dealing with empirical sampling, and a selection of papers making use of stochastic processes, including a bibliography prepared by Dr. David G. Kendall.

Richard Cook

Irrational Numbers. By Ivan Niven. John Wiley & Sons, New York 164 pages. $3.00.

Number Eleven of the Carus Mathematical Monographs, *Irrational Numbers* was published in August by the Mathematical Association of America. It will be distributed by John Wiley & Sons. The author is Ivan Niven, professor of mathematics at the University of Oregon.

This book provides an exposition of some central results on irrational, transcendental, and normal numbers. There is a complete treatment by elementary methods of the irrationality of the exponential, logarithmic, and trigonometric functions with rational arguments.

The approximation of irrational numbers by rationals, up to such results as the best possible approximation of Hurwitz, is also given with elementary techniques. The last third of the monograph treats normal and transcendental numbers, including the transcendence of π and its generalization in the Lindemann theorem, and the Gelfond-Schneider theorem.

Richard Cook

Digital Differential Analyzers. By George F. Forbes, 10117 Bartee Avenue, Pacoima, California, Paper cover, approximately 200 pages including appendices and nine page index. $7.50

An applications manual dealing with mathematics in the form specifically related to digital differential analyzers. A knowledge of differential and integral calculus is presupposed. Familiarity with more advanced mathematics and with a specific DDA is desirable but not essential.

The common functions are discussed in some detail, as well as servo techniques and limiting methods. Differentiation, the more advanced functions, simultaneous equations, and implicit algebraic functions are described in a more general way. Following a section on errors of the DDA, the more advanced sections of the text deal with complex functions, coordinate system transformations, conformal mapping, partial differential equations, and curve analysis.

Several illustrative problems are discussed. The DDA is treated both as a mathematical device and as an engineering and scientific tool.

This year International Business Machines estimates that 7,500 Mathematicians will be needed to man Computers now on order, of which about 1,500 should be Ph. D's. About 250 will graduate this year!

From *Aviation Week*, Sept. 17, 1956, p. 25

PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to sumbit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

PROPOSALS

285. *Proposed by Maj. H.S. Subba Rao, New Delhi, India.*

It is well known that given any two areas enclosed by closed curves in a plane there exists a straight line which bisects both the areas. Is it possible to construct such a line when the two areas are: a) a triangle and a parallelogram, b) two triangles?

286. *Proposed by Joseph Andruskiw, Seton Hall University.*

Evaluate

$$I = \int_0^{\pi/2} x\left\{\frac{\sin x}{1+\cos^2 x} + \frac{\cos x}{1+\sin^2 x}\right\} dx$$

287. *Proposed by Chih-yi Wang, University of Minnesota.*

Show that

$$\cot(\pi/11) - \cot(2\pi/11) + \cot(4\pi/11) - \cot(8\pi/11) + \cot(16\pi/11) = \sqrt{11}$$

288. *Proposed by Norman Anning, Alhambra, California.*

A particle moves on a Fermat Spiral, $r^2 = a^2\theta$, under a central force directed to the origin. Show that the law of force involves the seventh power of the radius in a denominator.

289. *Proposed by J.M. Howell, Los Angeles City College.*

Prove or disprove the following statement: The diagonal of a rectangular parallelopiped can be any odd integer other than one or five, and the edges relatively prime integers.

290. *Proposed by Leon Bankoff, Los Angeles, California.*

The hypotenuse AC of a right triangle ABC is divided into unequal segments x and y by the point of contact of the incircle. Given segments x and y: a) construct the triangle ABC and its incircle, b) without using the fourth proportional, upon x as a base, construct a rectangle equivalent to triangle ABC.

291. *Proposed by M.S.Klamkin, Polytechnic Institute of Brooklyn.*

Determine the minimum of

$$\frac{\sum_{r=1}^{s} x_r^s}{\prod_{r=1}^{s} x_r} \qquad \text{where } x_r > 0$$

Solutions

Probability of Convexity of Four Points

264. [March 1956] *Proposed by Norman Anning, Alhambra, California.*

Four points are thrown at random on a plane. What is the probability that they will be the vertices of a convex quadrilateral?

Solution by Huseyin Demir, Kandilli, Bolgesi, Turkey. The probability of three points to form a triangle is evidently 1. Let then ABC be the triangle formed by three of the points. For the points A, B, C, D to form a convex polygon it is necessary that the fourth point D be in the region of ABC where we have excircles. These regions of the plane correspond to the angles A, B, C of the triangle ABC. Hence

$$p = (A + B + C)/\text{whole plane} = \pi/2\pi = ½$$

Also solved by Richard K. Guy, University of Malaya, Singapore; John M. Howell, Los Angeles City College and the proposer.

A Volume of Revolution

265. [March 1956] *Proposed by Stephen Armstrong, Union College Schenectady, New York.*

What is the volume of revolution of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ revolved about the line $x = y$?

I. *Solution by Lawrence A. Ringenberg, Eastern Illinois State College.* Assume $a > b > 0$. Let S denote the sector with central angle $\pi/2$ which is bounded on the left by the lines $y = \pm x$ and on the right by the given ellipse. If (x,y) is a point in S, its distance from the line $y = x$ is $(x - y)/\sqrt{2}$. Since the desired volume V is twice the volume generated when S revolves about the line $y = x$, we may compute it as twice the double integral of $\sqrt{2}\pi(x - y)$ over S. Thus

$$V = 2\sqrt{2}\,\pi \iint_S (x - y)\, dS$$

Set

$$f(y) = \frac{a}{b}\sqrt{b^2 - y^2}, \qquad A = \frac{ab}{\sqrt{a^2 + b^2}}.$$

Then

$$V = 2\sqrt{2}\,\pi \int_{-A}^{0}\int_{-y}^{f(y)} (x - y)\, dx\, dy + 2\sqrt{2}\,\pi \int_{0}^{A}\int_{y}^{f(y)} (x - y)\, dx\, dy$$

and

$$V = \frac{4\sqrt{2}\,\pi a^3 b}{3\sqrt{a^2 + b^2}}.$$

Comment. If the ellipse is rotated through one-half a revolution, the volume V_0 generated may be computed as the double integral of $\sqrt{2}\pi(x - y)$ taken over that portion of the xy-plane which lies within the ellipse and below the line $y = x$. It may be interesting to note that V_0 could also be computed as follows. Let the lines $y = \pm x$ divide the interior of the ellipse into four quadrants S_1, S_2, S_3, S_4, numbered counterclockwise starting with $S_1 = S$, S of the preceding paragraph. Let V_i, $i = 1,2,3,4$, be the volume generated when S_i rotates one-half revolution about $y = x$. Then if V is the volume obtained in the foregoing solution and

$V' = \dfrac{4\sqrt{2}\,\pi ab^3}{3\sqrt{a^2 + b^2}}$, obtained from V by interchanging a and b, we have

$$V_1 = V_3 = V/4, \quad V_2 = V_4 = V'/4, \text{ and } V_0 = \frac{V + V'}{2} = \frac{2\sqrt{2}\pi ab\sqrt{a^2 + b^2}}{3}$$

II. *Solution by C.N. Mills, Augustana College, Sioux Falls, South Dakota*. There are two equal generating surfaces revolving about the axis $y = x$. The area of each surface is

$$\frac{ab}{2}\left[\pi - 2 \arcsin \frac{b}{\sqrt{a^2 + b^2}}\right]$$

The distance of the centroid of each surface from the axis is

$$\frac{2\sqrt{2}\, a^2}{3\sqrt{a^2 + b^2}}\left[\pi - 2 \arcsin \frac{b}{\sqrt{a^2 + b^2}}\right].$$

Applying the theorem that the volume of a solid of revolution is equal to the area of the generating surface times the distance through which the centroid moves, we obtain

$$V = \frac{4\sqrt{2}\, a^3 b\pi}{3\sqrt{a^2+b^2}}$$

Also solved by Huseyin Demir, Kandilli Bolgesi, Turkey; J.M.C. Hamilton, Los Angeles City College; Richard K. Guy, University of Malaya, Singapore; Sister M. Stephanie, Georgian Court College, New Jersey and the Proposer.

Points Inverse in a Circumcircle

266.[March 1956] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

If M and M' are points inverse to each other with respect to the circumcircle of a triangle ABC then prove that:

$$\angle BMC + \angle BM'C = 2\angle A$$
$$\angle CMA + \angle CM'A = 2\angle B$$
$$\angle AMB + \angle AM'B = 2\angle C$$

I. *Solution by Richard K. Guy, University of Malaya, Singapore.* In triangles COM and $M'OC$ angle O is common and as $OM \cdot OM' = OC^2$ we have $\frac{OC}{OM} = \frac{OM'}{OC}$. Hence the triangles are similar and $\measuredangle OM'C = \measuredangle OCM$. In the same way $\measuredangle OM'B = \measuredangle CBM$. Adding these to $\measuredangle OMC$ and $\measuredangle OMB$ we have $\measuredangle BMC + \measuredangle BM'C = \pi - \measuredangle COM + \pi - \measuredangle BOM = \measuredangle BOC = 2\measuredangle A$.

Simmlarly we have $\measuredangle CMA + \measuredangle CM'A = 2\measuredangle B$ and $\measuredangle AMB = \measuredangle AM'B = 2\measuredangle C$.

II. *Solution by Maimouna Edy, Hull, PQ, Canada.* Represent points A, B, C, M, M' by complex numbers z_1, z_2, z_3, z, z' respectively. Let parentheses represent cross ratios and the bars the complex conjugate. We then have:

$$(1)\qquad (z_1, z_2, z_3, z) = \overline{(z_1, z_2, z_3, z')}$$

This says that the homographic transformation which sends z_1, z_2, z_3 into 1, 0, ∞ respectively, that is the transformation of the given circle into the axis of reals, sends z and z' into two conjugate complex points.

Now equation (1) reads explicitly

$$\frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} = \overline{\frac{z' - z_2}{z' - z_3}} \cdot \overline{\frac{z_1 - z_3}{z_1 - z_2}}$$

Evidently

$$\frac{z_2 - z}{z_3 - z} \div \overline{\frac{z_2 - z'}{z_3 - z'}} = \overline{\frac{z_3 - z_1}{z_2 - z_1}} \div \frac{z_3 - z_1}{z_2 - z_1}.$$

Therefore

$$\arg\left(\frac{z_2 - z}{z_3 - z}\right) + \arg\left(\frac{z_2 - z'}{z_3 - z'}\right) = -2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = +2\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right)$$

This means that for oriented angles,

$$(\text{angle } \vec{MC}, \vec{MB}) + (\text{angle } \vec{M'C}, \vec{M'B}) = 2(\text{angle } \vec{AC}, \vec{AB}).$$

The oriented angles form an additive group isomorphic with the multiplicative group of the unit circle. In other words, we may take arbitrary measures of our angles and add them (mod 2π). The other two relations are proven similarly.

Bankoff's solution also noted the necessity for proper orientation of the angles.

Also solved by Leon Bankoff, Los Angeles, California; J.W. Clawson, Collegeville, Pennsylvania; and the proposer.

Limit of a Radical Sum

267. [March 1956] *Proposed by Alan Wayne, Cooper Union School of Engineering, New York.*

If $R_m = \sum_{k=1}^{m} \frac{\sqrt{k^n - (k-1)^n}}{\sqrt{m^{n+1}}}$, prove that $\lim_{m\to\infty} R_m = \frac{2\sqrt{n}}{n+1}$

I. *Solution by Paul Schillo, University of Buffalo.* Let $P(c,u,r)$ denote any polynomial in u of degree t, where $0 \le t \le r$ and where c is the coefficient of u^r. If p is a positive integer and $b \neq 0$, then letting $Q(1/m)$ denote a function f of m and k such that $mf(m,k)$ is bounded for all positive integer pairs (m,k), in which m is greater than the greatest integral root of $P(b,m,r+p) = 0$ and $k \le m$, it follows (by the fundamental theorem of the integral calculus) that

$$\lim_{m\to\infty} \sum_{k=1}^{m} \left[\frac{P(a,k,r)}{P(b,m,r+p)}\right]^{1/p} = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \left[\left(\frac{a}{b}\right)\left(\frac{k}{m}\right)^r + Q\left(\frac{1}{m}\right)\right]^{1/p}$$

$$= \int_0^1 \left[\left(\frac{a}{b}\right) x^r\right]^{1/p} dx = \left(\frac{p}{r+p}\right)\left(\frac{a}{b}\right)^{1/p}$$

In the given problem, $k^n - (k-1)^n = P(n, k, n-1)$ and $m^{n+1} = P(1, m, n+1)$, so $\lim_{m\to\infty} R_m = \left(\frac{2}{n-1+2}\right)\sqrt{\frac{n}{1}} = \frac{2\sqrt{n}}{n+1}$.

It might be added that the above limit holds if the polynomials have complex coefficients, since it is the line integral of an everywhere analytic function.

II. *Solution by the proposer.* Make use of the (u, v) rectangular plane coordinate system in which $u = x + iy$, $v = x - iy$, and $i = \sqrt{-1}$. In this (u, v) plane, the distance between the two points $P_1(u_1, v_1)$ and $P_2(u_2, v_2)$ is given by $\overline{P_1P_2} = (u_2 - u_1)^{1/2}(v_2 - v_1)^{1/2}$. Moreover, the length of the arc of any rectifiable curve $v = f(u)$ joining P_1 and P_2 is given by

$$\overline{P_1P_2} = \int_{u_1}^{u_2} (dv/du)^{1/2}\, du.$$

Now consider the curve $v = u^n$, with its points

$$P_k(k/m,\ k^n/m^n), \quad (k = 0, 1, 2, \ldots, m).$$

Clearly for $k > 0$, $\overline{P_{k-1}P_k} = \sqrt{k^n - (k-1)^n} \,/\, \sqrt{m^{n+1}}$. Then

$$R_m = \sum_{k=1}^{m} \overline{P_{k-1}P_k}\ .$$

Hence

$$\lim_{m\to\infty} R_m = \overline{P_oP_m} = \int_0^1 (nu^{n-1})^{1/2}\, du = 2\sqrt{n}/(n+1)$$

Similar limits of sequences of series can be evaluated by this "transformation" method.

Also solved by Maimouna Edy, Hull, P.Q. Canada and Chih-yi Wang, University of Minnesota.

Coaxal Circles

268. [March 1956] *Proposed by J.W. Clawson, Collegeville, Pennsylvania.*

Three coaxal circles, centers at A, B and P have the common points C and D. Any straight line is drawn through C cutting the circles again in the points L, M and N respectively. Prove that the ratio LN/NM equals the ratio AP/PB.

Solution by Chih-yi Wang, University of Minnesota. Let the feet of perpendiculars from A, B and P onto the line $LCMN$ be A', B' and P' respectively. Then we have

$$CP' = CN/2, \quad CB' = CM/2, \quad A'C = LC/2;$$

$$\frac{P'B'}{PB} = \frac{B'P'}{BP} = \frac{A'B'}{AB} = \frac{A'B' + B'P'}{AB + BP} = \frac{A'P'}{AP} \quad \text{or} \quad \frac{A'P'}{P'B'} = \frac{AP}{PB},$$

whence
$$\frac{LN}{NM} = \frac{LN/2}{NM/2} = \frac{A'C + CP'}{CB' + P'C'} = \frac{A'P'}{P'B'} = \frac{AP}{PB}$$

Remark: If the line passing through C is the line CD, L, M and N coincide and in this exceptional case the ratio LN/MN is of the form $0/0$.

Also solved by Huseyin Demir, Kandilli, Bolgesi, Turkey; Maimouna Edy, Hull, P.Q., Canada; Richard K. Guy, University of Malaya, Singapore; and the proposer.

The Product of Power Series

269. [March 1956] *Proposed by M.S. Klamkin, Polytechnic Institute of Brooklyn.*

Find the sum $\displaystyle\sum_{n=1}^{\infty}\left[\frac{n}{1!} + \frac{n-1}{2!} + \frac{n-2}{3!} + \dots + \frac{1}{n!}\right] x^n$

Solution by Maimouna Edy, Hull, P.Q., Canada.

$$\sum_{n=1}^{\infty}\left[\frac{n}{1!} + \frac{n-1}{2!} + \frac{n-2}{3!} + \dots + \frac{1}{n!}\right] x^n$$

$$=\left(\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}\right)\cdot\left(\sum_{n=0}^{\infty} nx^n\right) = \left(\frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!}\right)\left(x\cdot\sum_{n=0}^{\infty} nx^{n-1}\right)$$

$$=\left(\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}\right)\left(\sum_{n=0}^{\infty} nx^{n-1}\right) = e^{x-1}(1-x)^{-2}.$$

This result is valid for $|x| < 1$.

Also solved by L. Carlitz, Duke University; V.D. Gokhale, Atlanta University, Georgia; John M. Howell, Los Angeles City College; Alice A. Huffman, University of California at Riverside; Lawrence A. Ringenberg, Eastern Illinois State College; Paul Schillo, University of Buffalo; Chih-yi Wang, University of Minnesota and the proposer.

Circles In A Crescent

270. [March 1956] *Proposed by Leon Bankoff, Los Angeles, California.*

A maximum circle is inscribed in a crescent formed by a semicircle and a quadrant of a circle. Find a general expression for the radii of consecutively tangent circles touching the sides of the crescent, the first touching the maximum circle, the second touching the first and so on.

Solution by Huseyin Demir, Kandilli, Bolgesi, Turkey. Let the given circles (0), (0′) intersect each other at A and B, and let the center and radius of the nth circle be denoted by (0_n), r_n respectively.

We invert the figure with center at A, $k^2 = AB^2 = 4R^2$ being the power. Under the inversion, (0) is inverted into its tangent line BH_0, and (0′) into the line $B0'$, forming an angle of $2\alpha = 45°$. The circles (Ω_i), inverse of (0_i), from a series of tangent circles inscribed in the above angle. Let (Ω_n) touch BH_0 at H_n. Then we may easily find that the radius $\rho_n = \Omega_n H_n$ of (Ω_n) is given by

$$\rho_n = \left(\frac{1 - \sin\alpha}{1 + \sin\alpha}\right)^n \rho_0$$

where

$$\rho_0 = \Omega_0 H_0 = BH_0 \tan\alpha = 2R \tan\alpha.$$

Drawing the common tangent AT_nT_n' to the inverse circles (0_n), (Ω_n) we write

$$r_n = AT_n\ \rho_n/AT_n' = AT_n \cdot AT_n' \rho_n/AT_n'^2 = k^2\rho_n/(A\Omega_n^2 - n^2)$$

Denoting the projection of Ω_n on AB by K_n we have

$$A\Omega_n^2 - \rho_n^2 = AK_n^2 + K_n\Omega_n^2 - \rho_n^2$$
$$= (2R + \rho_n)^2 + BH_n^2 - \rho_n^2$$
$$= 4R^2 + 4R\rho_n + (\rho_n \cot\alpha)^2$$
$$r_n = \frac{R^2\rho_n \tan\alpha}{4R^2 + 4R\rho_n + \rho_n^2 \cot^2\alpha}$$

Substituting the value of ρ_n in the above expression we arrive at the desired result, namely

$$r_n = \frac{R}{1 + \tfrac{1}{2}\left[(1 + \sin \pi/8)^{2n} + (1 - \sin \pi/8)^{2n}\right] \sec^{2n} \pi/8 \cot \pi/8}$$

Also solved by J.W. Clawson, Collegeville, Pennsylvania and the proposer.

TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

T 24. A certain physical society is planning a ballot for the election of three officers. There being 3, 4, and 5, candidates for the three offices, respectively. There is a rule in effect (in order to eliminate the ordering of the candidates on the ballot as a possible influence on the election) that for each office, each candidate must appear in each position the same number of times as any other candidate. What is the smallest number of different ballots necessary? [*Submitted by M.S. Klamkin*]

T 25. Given the sequence 1, 1, 2, 3, 5, 8, ... where $a_i = a_{i-1} + a_{i-2}$ find an expression for the nth term, a_n, not in terms of a recursion formula. Generalize the sequence so that a_1 and a_2 may be any two numbers. [*Submitted by Ben B. Bowen*].

SOLUTIONS

S 25. Off hand one would say $3 \cdot 4 \cdot 5 = 60$. However, if one adds two fictitious names to the group of three and one fictitious name to the group of four, then only five different ballots are necessary. Not only will this method reduce the printing costs, but it will also give statistics on whether or not members vote by relative order or not.

S 24. Consider the following sequence which has the same recursive property as the given sequence:

$$\text{if} \quad (1)\begin{cases} x^2 = x + 1 \\ y^2 = y + 1 \end{cases} \quad \text{then} \quad x^2 - y^2 = x - y$$

$$\text{and} \quad \begin{cases} x^3 = x^2 + x \\ y^3 = y^2 + y \end{cases} \quad \text{then} \quad x^3 - y^3 = x^2 - y^2 + x - y$$

and so on. For a non-trivial solution we set $x \neq y$. Solving (1) we have $x = \dfrac{1+\sqrt{5}}{2}$ and $y = \dfrac{1-\sqrt{5}}{2}$. Now let $a_1 = \dfrac{x-y}{\sqrt{5}} = 1$;

$a_2 = \dfrac{x^2 - y^2}{\sqrt{5}} = 1, \quad a_3 = \dfrac{x^3 - y^3}{\sqrt{5}} = 2, \ \ldots, \ a_n = \dfrac{x^n - y^n}{\sqrt{5}}$. To generalize the sequence, let $u_1 = k_1$ and $u_2 = k_2$, $u_3 = u_1 + u_2 = k_1 + k_2$, ... $u_n = u_{n-1} + u_{n-2}$. Observe that $u_n = u_2 a_{n-1} + u_1 a_{n-2}$ where the a_i are are defined above as $a_i = \dfrac{x^i - y^i}{\sqrt{5}}$ and $x = \dfrac{1+\sqrt{5}}{2}$, $y = \dfrac{1-\sqrt{5}}{2}$.

The theorem is true for $n = 2$ and 3 since $a_2 = a_1 = 1$. Assume it true for $n = k$ and $k = 1$, then

$$u_k = u_2 a_{k-1} + u_1 a_{k-2}$$
$$u_{k+1} = u_2 a_k + u_1 a_{k-1}$$

adding $u_{k+2} = u_2 a_{k-1}$ which satisfies the theorem.

Therefore

$$u_n = k_2 \frac{x^{n-1} - y^{n-1}}{\sqrt{5}} + k_1 \frac{x^{n-2} - y^{n-2}}{\sqrt{5}} .$$

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and source, if known.

Q 183. What are the ratios of the volumes and surfaces of the inner "half" to the outer "half" of the torus generated by revolving the $x^2 + (y - a)^2 = r^2$, $a > r$ [*Submitted by V.C. Harris*]

Q 184. Simplify

$$I = \frac{\sin x + \sin 2x + \sin 3x + \dots + \sin nx}{\cos x + \cos 2x + \cos 3x + \dots + \cos nx} .$$

[*Submitted by M.S. Klamkin*]

Q 185. Prove that

$$\sum_{r=0}^{k} (-1)^r \binom{n+r}{s} \binom{k}{r} = 0 \qquad \text{where } s \le k - 1$$

[*Submitted by M.S. Klamkin*]

ANSWERS

A 183. Using Pappus' Theorem we have:

$$\frac{V_i}{V_0} = \frac{2\pi \bar{y}_i A}{2\pi \bar{y}_0 A} = \frac{\bar{y}_i}{\bar{y}_0} = \frac{a - \frac{4r}{3\pi}}{a + \frac{4r}{3\pi}} = \frac{3\pi a - 4r}{3\pi a + 4r} \quad \text{and} \quad \frac{S_i}{S_0} = \frac{2\pi y_i s}{2\pi y_0 s} = \frac{\bar{y}_i}{\bar{y}_0} = \frac{a\pi - 2r}{a\pi + 2r}$$

where the quantities with the subscript i refer to the inner part and subscript 0 refer to the outer part of the torus.

A 184.

$$\sum_{1}^{n} \sin nx = \frac{\sin \frac{n-1}{2} x \sin \frac{n}{2} x}{\sin \frac{x}{2}}$$

and

$$\sum_{1}^{n} \cos nx = \frac{\cos \frac{n-1}{2} x \sin \frac{n}{2} x}{\sin \frac{x}{2}}$$

Therefore

$$I = \tan \frac{n-1}{2} x$$

A 185.

$$x^n(1-x)^k = \sum_{r=0}^{k} (-1)^r \binom{k}{r} x^{n+r}$$

Since $D^s(x^n)(1-x)^k \big|_{x=1} = 0$ the result follows.